The Asteroid Surveying Problem and Other Puzzles

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ABSTRACT

We consider two variants of the well-known "sailor in the fog" puzzle. The first version (the "asteroid surveying" problem) is set in three dimensions and asks for the shortest curve that starts at the origin and intersects all planes at unit distance from the origin. Several possible solutions are suggested in the video, including a curve of length less than 12.08. The second version (the "river shore" problem) asks for the shortest curve in the plane that has unit width. A solution of length $2.2782\ldots$ is described, which we have proved to be optimal.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*geometrical problems and computations*

General Terms

Theory

Keywords

curves, geometric constants

1. THE "ASTEROID SURVEYING" PROBLEM

In a popular puzzle from recreational mathematics [2, 3, 5], a sailor rows a mile out to sea, throws an anchor and a fishing line, and promptly falls asleep. By the time he wakes up, a dense fog has surrounded him. Knowing the distance to the shore but not knowing the direction, he wants to devise a path that is guaranteed to reach shore and that minimizes the distance travelled in the worst case. In other words, he would like to find the shortest curve that starts at the origin and intersects all lines at distance 1 from the origin. This problem was solved by Isbell in 1957 [3], and the optimal curve was discovered to have length $\sqrt{3}+7\pi/6+1 = 6.3972\ldots$

Puzzles of this kind have gained interest from researchers in computer science, particularly since the introduction of competitive analysis for robot navigation problems. One natural question, posed recently at CCCG'02 [1], concerns the three-dimensional generalization:

> What is the shortest curve in $\mathbb{R}^3$ that starts at the origin and intersects all planes at distance 1 from the origin?

This precise question arises in the following contrived (but amusing) setting. Imagine that a spaceship has already landed on the surface of a spherical asteroid, of unit radius. What is the shortest path that the spaceship can take in order to survey the entire surface of the asteroid? A point on the surface is visible from some point on the path if and only if the tangent plane through the point intersects the path. Insisting that the path starts at the boundary of the sphere rather than the origin only subtracts 1 from the optimal path length. So, the two questions are indeed equivalent.

A curve intersects all planes at distance 1 from the origin if and only if the curve's convex hull contains the unit sphere centered at the origin. Given any curve, we can compute the largest sphere centered at the origin inscribed in the convex hull and then scale coordinates by the reciprocal of the radius of this sphere to obtain a feasible solution.

In the video, we first consider a few simple polygonal curves, for example, through the vertices of a cube (with length $14+\sqrt{3} = 15.7320\ldots$), through the vertices of a regular tetrahedron (with length $6\sqrt{6}+3 = 17.6969\ldots$), and through the vertices of a regular octahedron (with length $5\sqrt{6}+\sqrt{3} = 13.9794\ldots$).

We then present the best curve that we have found currently through experimentation. It belongs to a family of spirals of the form

$$\{((1-at^2)\sin(b\pi t),\ (1-at^2)\cos(b\pi t),\ ct) \mid -1 \le t \le 1\},$$

with line segments attached from the top point to an extra point (x_0, y_0, z_0) and the bottom point to a symmetric point $(-x_0, -y_0, -z_0)$. The origin is implicitly connected to (x_0, y_0, z_0). See Figure 1 (left). (For clarity sake, the portion of the path from the origin is not drawn in the figure or the video.)

Intuitively, the $1 - at^2$ factor fattens the spiral near the middle, while the extra two line segments short-cut the curve near the top and bottom. A guided computer search suggests the following choice of parameters: $a = 0.4$, $b = 1.18$, $c = 1.12$, $x_0 = -0.37$, $y_0 = -0.199$, $z_0 = 1.24$. After scaling by a factor of roughly 1.363, this yields a feasible solution of length less than 12.08 (or in the asteroid surveying setting, 11.08).

2. THE "RIVER SHORE" PROBLEM

We next return to two dimensions and consider a related puzzle posed by Ogilvy [4]: starting at an unknown point

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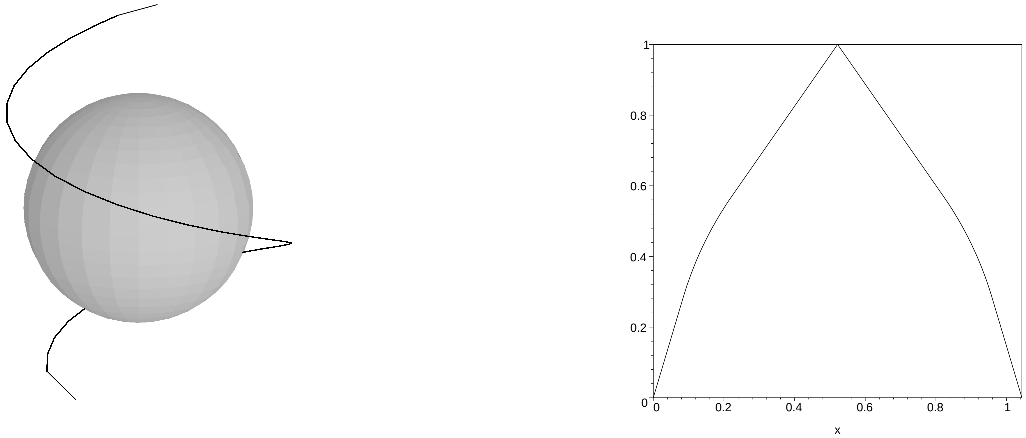

Figure 1: Our best "asteroid surveying" curve (left) and the optimal "river shore" curve (right).

inside a river of width 1, what is the shortest path that is guaranteed to reach one of the two shores of the river?

With a little thought, the question is seen to be equivalent to the following:

What is the shortest curve in $\mathbb{R}^2$ that has width 1?

(Recall that the width of a curve is defined as the width of the minimum strip enclosing the curve.) Notice that a solution to the sailor-in-the-fog puzzle, scaled by a half, has width at least 1, but the converse is not necessarily true.

For closed curves, we can take the circle of diameter 1, which has perimeter π. Interestingly, there are infinitely many closed curves (so-called *curves of constant width*) with the property that the width of the minimum enclosing strip along every direction is always 1, and surprisingly, all of them have the same perimeter π (this is known as *Barbier's theorem*).

For open curves, we can do better. For example, a V-shape formed by the vertices of an equilateral triangle already gives length $4/\sqrt{3} = 2.3094\ldots$

Better still, we consider a family of curves formed by the concatenation of a line segment, a circular arc, and a line segment, together with their reflection. We determine the best parameters using calculus and find a curve of length

$$2\left(\frac{x}{2} + \arccos\frac{4x}{x^2+4} - \arccos\frac{1}{x} + \sqrt{x^2-1}\right) = 2.2782\ldots$$

for $x = 2\sqrt{z}$, where $z = 0.2722\ldots$ is a root of the cubic $3z^3 + 9z^2 + z - 1$. See Figure 1 (right). This curve is in fact optimal; the main ideas behind the proof are sketched in the video.

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3. REFERENCES

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