Basics of Relation Algebra

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Plan

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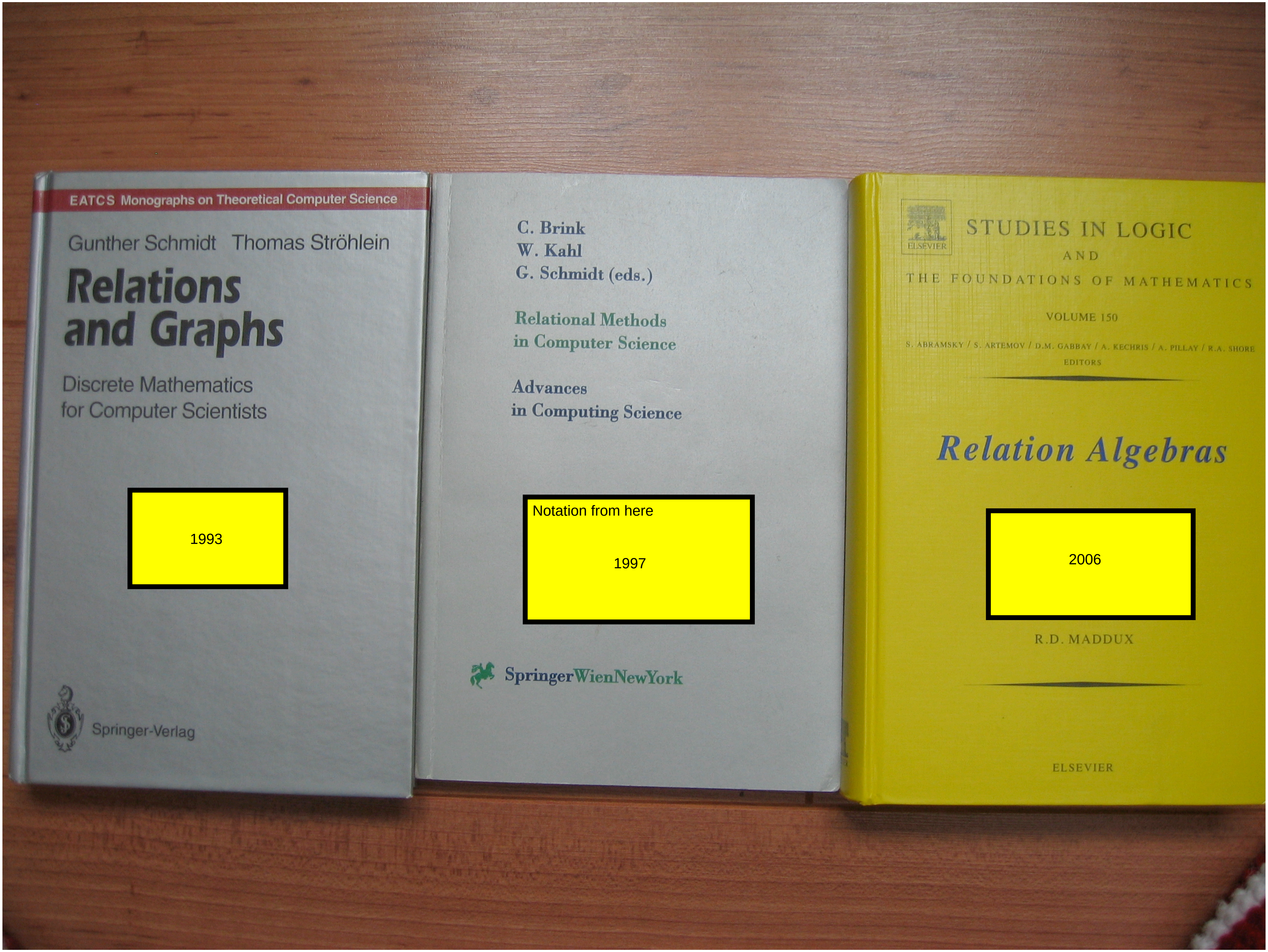
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VOLUME 150
S. ABRAMSKY / S. ARTEMOV / D.M. GABBAY / A. KECHRIS / A. PILLAY / R.A. SHORE
EDITORS
Relation Algebras
2006
R.D. MADDUX
ELSEVIER

1 Relations on a set

1. A relation on a set S is a subset of $S \times S$

2. The set of relations on S is $\mathcal{P}(S \times S)$

3. Notation: $sRt \;\Leftrightarrow\; (s,t) \in R, \qquad V = S \times S$

4. Operations on relations

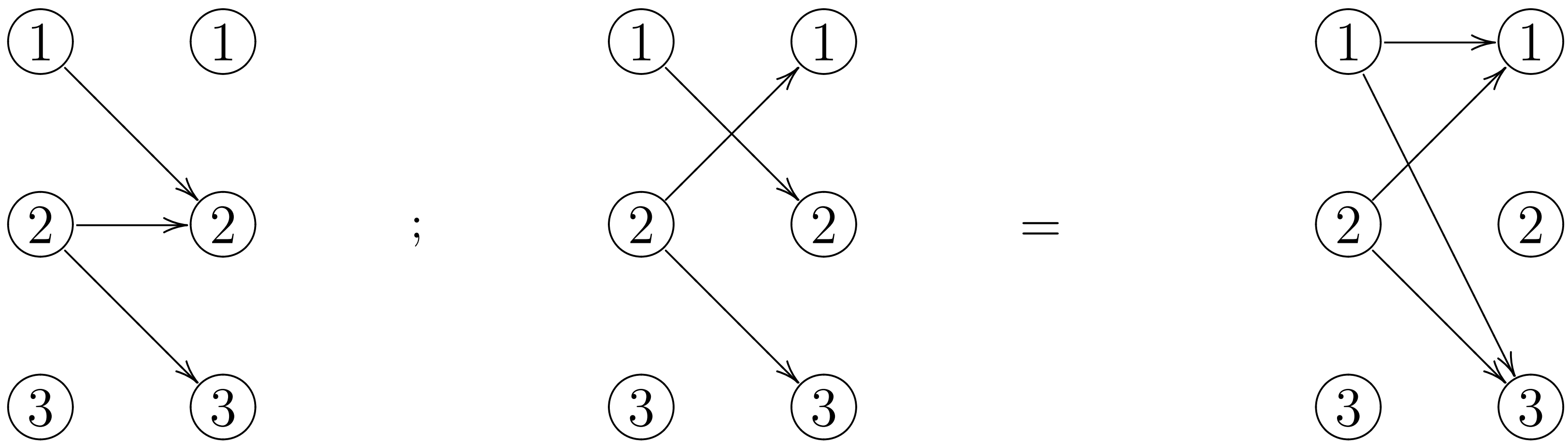
   - Set-theoretical operations: $\cup$, $\cap$, $\overline{}$, $\emptyset$, V
   - Relational operations
     - Composition (relative product): $sQ\,;Ru \Leftrightarrow (\exists t : sQt \wedge tRu)$
     - Converse: $sR^{\smile}t \Leftrightarrow tRs$
     - Identity relation: $sIt \Leftrightarrow s = t$

Representations of relations: sets of ordered pairs, graphs, matrices

Let $S \stackrel{\text{def}}{=} \{1, 2, 3\}$.

$$Q \quad ; \quad R \quad = \quad Q \,;\, R$$

$$\{(1,2),(2,2),(2,3)\} \quad ; \quad \{(1,2),(2,1),(2,3)\} \quad = \quad \{(1,1),(1,3),(2,1),(2,3)\}$$



$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Relations on a set satisfy many laws

Increasing priority: $(\cup, \cap)$, ;, $\breve{}$

- $Q \cup R = R \cup Q$
- $P \cap (Q \cap R) = (P \cap Q) \cap R$
- $\overline{(Q \cup R)} = \overline{Q} \cap \overline{R}$
- $I ; R = R$
- $P ;(Q \cup R) = P ; Q \cup P ; R$
- $P ;(Q \cap R) \subseteq (P ; Q) ; R$
- $(Q ; R)^{\smile} = R^{\smile} ; Q^{\smile}$
- $R \neq \emptyset \;\Rightarrow\; V ; R ; V = V$
- $P ; Q \subseteq R \;\Leftrightarrow\; P^{\smile} ; \overline{R} \subseteq \overline{Q} \;\Leftrightarrow\; \overline{R} ; Q^{\smile} \subseteq \overline{P}$
- $P ; Q \cap R \subseteq (P \cap R ; Q^{\smile}) ;(Q \cap P^{\smile} ; R)$
- $\ldots$

2 Relation algebra

Relation algebra (RA): Aims at "characterizing" relations on a set by means of simple equational axioms. It is a structure

$$\mathcal{A} = \langle A, \sqcup, \overline{}, ;, \smile, \mathbb{I} \rangle$$

such that

$$\left.\begin{array}{ll} (1) & Q \sqcup R = R \sqcup Q \\ (2) & P \sqcup (Q \sqcup R) = (P \sqcup Q) \sqcup R \\ (3) & \overline{\overline{Q} \sqcup \overline{R}} \sqcup \overline{\overline{Q} \sqcup R} = Q \end{array}\right\} \text{ Boolean algebra axioms}$$

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Another axiomatisation of Boolean algebra:

Add $\sqcap, \perp\!\!\!\perp, \top\!\!\!\top$ to the signature, replace Huntington's axiom (3) by

$Q \sqcap R = R \sqcap Q$

$P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R$

$Q \sqcup (Q \sqcap R) = Q$

$Q \sqcap (Q \sqcup R) = Q$

$P \sqcup (Q \sqcap R) = (P \sqcup Q) \sqcap (P \sqcup R)$

$R \sqcup \perp\!\!\!\perp = R$

$R \sqcap \top\!\!\!\top = R$

$R \sqcup \overline{R} = \top\!\!\!\top$

$R \sqcap \overline{R} = \perp\!\!\!\perp$

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\end{array}\right\} \text{Boolean algebra axioms}$$

$$\begin{array}{rl}
(4) & P ; (Q ; R) = (P ; Q) ; R \\
(5) & (P \sqcup Q) ; R = P ; R \sqcup Q ; R \\
(6) & R ; \mathbb{I} = R \\
(7) & R^{\smile\smile} = R \\
(8) & (Q \sqcup R)^{\smile} = Q^{\smile} \sqcup R^{\smile} \\
(9) & (Q ; R)^{\smile} = R^{\smile} ; Q^{\smile} \\
(10) & Q^{\smile} ; \overline{Q ; R} \sqcup \overline{R} = \overline{R}
\end{array}$$

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$$\begin{array}{ll} (4) & P;(Q;R) = (P;Q);R \\ (5) & (P \sqcup Q);R = P;R \sqcup Q;R \\ (6) & R;\mathbb{I} = R \\ (7) & R^{\smile\smile} = R \\ (8) & (Q \sqcup R)^{\smile} = Q^{\smile} \sqcup R^{\smile} \\ (9) & (Q;R)^{\smile} = R^{\smile};Q^{\smile} \\ (10) & Q^{\smile};\overline{Q;R} \sqcup \overline{R} = \overline{R} \end{array}$$

Ordering $\sqsubseteq$

Define

$$Q \sqsubseteq R \;\Leftrightarrow\; Q \sqcup R = R.$$

Then (10) can be written

$$Q^{\smile};\overline{Q;R} \sqsubseteq \overline{R}.$$

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$$\begin{array}{ll} (4) & P ;(Q ; R) = (P ; Q) ; R \\ (5) & (P \sqcup Q) ; R = P ; R \sqcup Q ; R \\ (6) & R ; \mathbb{I} = R \\ (7) & R^{\smile\smile} = R \\ (8) & (Q \sqcup R)^{\smile} = Q^{\smile} \sqcup R^{\smile} \\ (9) & (Q ; R)^{\smile} = R^{\smile} ; Q^{\smile} \\ (10) & Q^{\smile} ; \overline{Q ; R} \sqcup \overline{R} = \overline{R} \end{array}$$

Derived operators $\sqcap, \perp\!\!\!\perp, \top\!\!\!\top$

$$Q \sqcap R = \overline{\overline{Q} \sqcup \overline{R}}$$

$$\perp\!\!\!\perp = \overline{\mathbb{I}} \sqcap \mathbb{I}$$

$$\top\!\!\!\top = \overline{\mathbb{I}} \sqcup \mathbb{I}$$

Laws that can be proved from the axioms

Relation algebra	Corresponding laws, relations on sets
$P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R$	$P \cap (Q \cap R) = (P \cap Q) \cap R$
$\overline{(Q \sqcup R)} = \overline{Q} \sqcap \overline{R}$	$\overline{(Q \cup R)} = \overline{Q} \cap \overline{R}$
$\mathbb{I} ; R = R$	$I ; R = R$
$\mathbb{I}^{\smile} = \mathbb{I}$	$I^{\smile} = I$
$\mathbb{T}^{\smile} = \mathbb{T}$	$V^{\smile} = V$
$\mathbb{T} ; \mathbb{T} = \mathbb{T}$	$V ; V = V$
$P ;(Q \sqcup R) = P ; Q \sqcup P ; R$	$P ;(Q \cup R) = P ; Q \cup P ; R$
$P ; Q \sqsubseteq R \iff P^{\smile} ; \overline{R} \sqsubseteq \overline{Q}$	$P ; Q \subseteq R \iff P^{\smile} ; \overline{R} \subseteq \overline{Q}$
$\iff \overline{R} ; Q^{\smile} \sqsubseteq \overline{P}$	$\iff \overline{R} ; Q^{\smile} \subseteq \overline{P}$
$P ; Q \sqcap R \sqsubseteq (P \sqcap R ; Q^{\smile}) ;(Q \sqcap P^{\smile} ; R)$	$P ; Q \cap R \subseteq (P \cap R ; Q^{\smile}) ;(Q \cap P^{\smile} ; R)$
???	$R \neq \emptyset \Rightarrow V ; R ; V = V$
$\vdots \quad \vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots \quad \vdots \quad \vdots$

Properties of the equational axiomatisation

Because RAs are defined by equations, the class of RAs is a **variety**: it is closed under **products**, **homomorphic images** and **subalgebras**.

Example. Consider the relations on $S_2 \stackrel{\text{def}}{=} \{1, 2\}$ and $S_3 \stackrel{\text{def}}{=} \{1, 2, 3\}$ or, equivalently, the subsets of

$$V_2 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad V_3 \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

1. **Products** The set of pairs of relations

   $$A_{2,3} = \{(R_2, R_3) \mid R_2 \subseteq V_2 \;\wedge\; R_3 \subseteq V_3\}$$

   is an RA with identity (I_2, I_3). Operations are defined pointwise. E.g., $(Q_2, Q_3) ; (R_2, R_3) = (Q_2 ; R_2, Q_3 ; R_3)$ and $(R_2, R_3)^{\smile} = (R_2^{\smile}, R_3^{\smile})$. The top relation is (V_2, V_3) or, on an isomorphic matrix form,

   $$V_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

2. **Homomorphic images** Let $f : A_{2,3} \to \mathcal{P}(S_2 \times S_2)$ be defined by

$$f((R_2, R_3)) = R_2.$$

Then f is a homomorphism.

An RA homomorphism is defined by the following properties.

$$
\begin{array}{rcl|rcl}
\multicolumn{3}{c|}{f : \mathcal{A} \to \mathcal{B}} & \multicolumn{3}{c}{f : A_{2,3} \to \mathcal{P}(S_2 \times S_2)} \\
\hline
f(Q \sqcup_{\mathcal{A}} R) & = & f(Q) \sqcup_{\mathcal{B}} f(R) & f((Q_2, Q_3) \cup (R_2, R_3)) & = & Q_2 \cup R_2 \\
f(\overline{R}^{\mathcal{A}}) & = & \overline{f(R)}^{\mathcal{B}} & f(\overline{(R_2, R_3)}) & = & \overline{R_2} \\
f(Q \,;_{\mathcal{A}} R) & = & f(Q) \,;_{\mathcal{B}} f(R) & f((Q_2, Q_3) \,; (R_2, R_3)) & = & Q_2 \,; R_2 \\
f(R^{\smile \mathcal{A}}) & = & (f(R))^{\smile \mathcal{B}} & f((R_2, R_3)^{\smile}) & = & (R_2)^{\smile} \\
f(\mathbb{I}_{\mathcal{A}}) & = & \mathbb{I}_{\mathcal{B}} & f((I_2, I_3)) & = & I_2
\end{array}
$$

The image of an RA homomorphism is an RA.

3. Subalgebras

- An RA

$$\mathcal{B} = \langle B, \sqcup, \overline{}, ;, \breve{}, \mathbb{I} \rangle$$

  is a subalgebra of an RA

$$\mathcal{A} = \langle A, \sqcup, \overline{}, ;, \breve{}, \mathbb{I} \rangle$$

  if $A \subseteq B$ (note: the operations are the same).

- For instance,

$$\langle \{ \perp\!\!\!\perp, \mathbb{I}, \overline{\mathbb{I}}, \top\!\!\!\top \}, \sqcup, \overline{}, ;, \breve{}, \mathbb{I} \rangle$$

  is a subalgebra of every RA with at least 4 elements.

- Given $B \subseteq A$, a subalgebra can be generated by closing B under the operations of $\mathcal{A}$.

Models of the axioms

1. Relations on a set S where **the universal relation V is an equivalence relation**. For instance, all relations included in

$$V_{2,3} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

form a relation algebra. Now let

$$R_{2,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and consider the composition

$$V_{2,3} \ ; \ R_{2,3} \ ; \ V_{2,3} \, .$$

$$
\begin{aligned}
V_{2,3} \; ; \; R_{2,3} \; ; \; V_{2,3} &= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} ; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&\neq V_{2,3}
\end{aligned}
$$

Thus, RAs do not in general satisfy the **Tarski rule**

$$R \neq \bot\!\!\!\bot \;\Leftrightarrow\; \top\!\!\!\top ; R ; \top\!\!\!\top = \top\!\!\!\top .$$

Those that do are **simple RAs** (with only two homomorphic images: themselves and the trivial RA with one element; they are not closed under products). For concrete relations on a set S, they are those with

$$V = S \times S.$$

Models of the axioms

2. Let $\mathcal{A} = \langle A, \sqcup, \overline{}, ;, \smile, \mathbb{I}\rangle$ be an RA. Let $\mathcal{M}_n$ be the set of $n \times n$ matrices with elements of A as entries. Define the following (red) operations on $\mathcal{M}_n$.

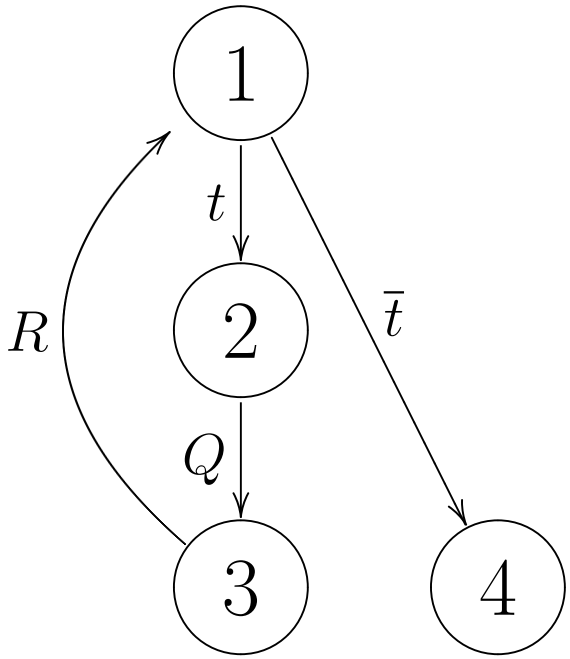
Operation	Definition
$\sqcup$	$(\mathbf{M} \sqcup \mathbf{N})[i,j] \stackrel{\text{def}}{=} \mathbf{M}[i,j] \sqcup \mathbf{N}[i,j]$
$\overline{}$	$\overline{\mathbf{M}}[i,j] \stackrel{\text{def}}{=} \overline{\mathbf{M}[i,j]}$
$;$	$(\mathbf{M} ; \mathbf{N})[i,j] \stackrel{\text{def}}{=} (\bigsqcup k : \mathbf{M}[i,k] ; \mathbf{N}[k,j])$
$\smile$	$\mathbf{M}^{\smile}[i,j] \stackrel{\text{def}}{=} (\mathbf{M}[j,i])^{\smile}$
$\mathbb{I}$	$\mathbb{I}[i,j] \stackrel{\text{def}}{=} \begin{cases} \mathbb{I} & \text{if } i = j \\ \perp\!\!\!\perp & \text{if } i \neq j \end{cases}$

Then

$$\langle \mathcal{M}_n, \sqcup, \overline{}, ;, \smile, \mathbb{I}\rangle$$

is an RA.

This model can be used for the description of programs.



$$\begin{pmatrix} \perp\!\!\!\perp & t & \perp\!\!\!\perp & \bar{t} \\ \perp\!\!\!\perp & \perp\!\!\!\perp & Q & \perp\!\!\!\perp \\ R & \perp\!\!\!\perp & \perp\!\!\!\perp & \perp\!\!\!\perp \\ \perp\!\!\!\perp & \perp\!\!\!\perp & \perp\!\!\!\perp & \perp\!\!\!\perp \end{pmatrix}$$

The matrix (graph) represents the control structure. The entries of the matrix (labels of the graph) are relations describing how the state changes by a transition.

Expressing properties of relations

Property	Definition	Set-theoretical expression
R total	$R\,;\mathbb{T} = \mathbb{T}$	$(\forall x : (\exists y : xRy))$
	$\mathbb{I} \sqsubseteq R\,;R^{\smile}$	
	$\overline{R} \sqsubseteq R\,;\overline{\mathbb{I}}$	
R functional	$R^{\smile}\,;R \sqsubseteq \mathbb{I}$	$(\forall x, y, z : xRy \wedge xRz \Rightarrow y = z)$
(or univalent)	$R\,;\overline{\mathbb{I}} \sqsubseteq \overline{R}$	
R reflexive	$\mathbb{I} \sqsubseteq R$	$(\forall x : xRx)$
R antisymmetric	$R \sqcap R^{\smile} = \mathbb{I}$	$(\forall x, y : xRy \wedge yRx \Rightarrow x = y)$
R transitive	$R\,;R \sqsubseteq R$	$(\forall x, y, z : xRy \wedge yRz \Rightarrow xRz)$
⋮	⋮	⋮

Using the relational instead of the set-theoretical definitions leads to equational proofs that are more compact and easier to verify.

Other properties

Property	Definition	Expression
R surjective	$R^{\smile}$ total	$R^{\smile} ; \mathbb{T} = \mathbb{T} \quad (\text{or } \mathbb{T} ; R = \mathbb{T})$ $\mathbb{I} \sqsubseteq R^{\smile} ; R$ $\overline{R^{\smile}} \sqsubseteq R^{\smile} ; \overline{\mathbb{I}} \quad (\text{or } \overline{R} \sqsubseteq \overline{\mathbb{I}} ; R)$
R injective	$R^{\smile}$ functional	$R ; R^{\smile} \sqsubseteq \mathbb{I}$ $R^{\smile} ; \overline{\mathbb{I}} \sqsubseteq \overline{R^{\smile}} \quad (\text{or } \overline{\mathbb{I}} ; R \sqsubseteq \overline{R})$
R mapping (or total function)	R total and functional	$R ; \overline{\mathbb{I}} = \overline{R}$
R bijective	R injective and surjective	$\overline{\mathbb{I}} ; R = \overline{R}$
R bijective mapping		$R^{\smile} ; R = \mathbb{I} \quad \text{and} \quad R ; R^{\smile} = \mathbb{I}$
R partial order	R reflexive, antisymmetric and transitive	

Representing subsets

There are three equivalent ways to represent subsets by relations.

1. Vectors: relations of the form $R ; \mathbb{T}$.
2. Covectors: relations of the form $\mathbb{T} ; R$.
3. Subidentities (tests, types): relations t such that $t \sqsubseteq \mathbb{I}$.

Example: representation of the subset $\{1, 2\}$ of $\{1, 2, 3\}$

1. Vector $\{(1,1),(1,2),(1,3),(2,1),(2,1),(2,3)\}$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
2. Covector $\{(1,1),(2,1),(3,1),(1,2),(2,2),(3,2)\}$ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$
3. Subidentity $\{(1,1),(2,2)\}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Vectors, covectors, tests form a Boolean algebra

Let $\mathcal{A} = \langle A, \sqcup, \sqcap, \overline{}, ;, \breve{}, \bot\!\!\!\bot, \top\!\!\!\top, \mathbb{I} \rangle$ be an RA
($\sqcap, \bot\!\!\!\bot, \top\!\!\!\top$ added to the signature for simplicity).

1. Vectors: $\langle \{R ; \top\!\!\!\top \mid R \in A\}, \sqcup, \sqcap, \overline{}, \bot\!\!\!\bot, \top\!\!\!\top \rangle$ is a BA.

2. Covectors: $\langle \{\top\!\!\!\top ; R \mid R \in A\}, \sqcup, \sqcap, \overline{}, \bot\!\!\!\bot, \top\!\!\!\top \rangle$ is a BA.

3. Tests: For $t \sqsubseteq \mathbb{I}$, define $\neg t \overset{\text{def}}{=} \bar{t} \sqcap \mathbb{I}$.

$$\langle \{t \mid t \sqsubseteq \mathbb{I}\}, \sqcup, \sqcap, \neg, \bot\!\!\!\bot, \mathbb{I} \rangle \quad \text{is a BA.}$$

   For this BA, $s \sqcap t = s ; t$. Tests occur in structures without $\sqcap$ and $\top\!\!\!\top$, like Kleene algebra with tests.

Correspondence between vectors, covectors and tests

1. Vector to covector: $R\,;\mathbb{T} \mapsto \mathbb{T}\,;R^{\smile}$
2. Covector to vector: $\mathbb{T}\,;R \mapsto R^{\smile}\,;\mathbb{T}$
3. Test to vector: $t \mapsto t\,;\mathbb{T}$
4. Vector to test: $R\,;\mathbb{T} \mapsto \mathbb{I} \sqcap R\,;\mathbb{T}$

Prerestriction and postrestriction

	prerestriction	postrestriction
vector	$R\,;\mathbb{T} \sqcap Q$	
covector		$\mathbb{T}\,;R \sqcap Q$
test	$t\,;Q$	$Q\,;t$

Representability

1. An RA is **representable** if it is isomorphic to a subalgebra of a concrete relation algebra (one that consists of the subsets of an equivalence relation).

2. There exist nonrepresentable RAs (even finite ones).

3. The class of representable RAs can only be axiomatised with an infinite number of axioms.

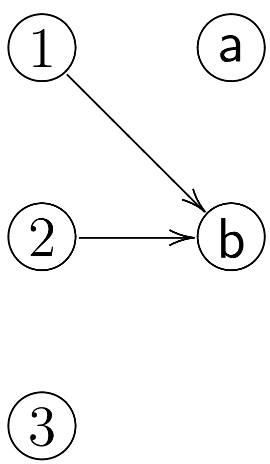
3 Heterogeneous relation algebra

Relations between different sets

Let $S = \{1, 2, 3\}$, $T = \{\mathsf{a}, \mathsf{b}\}$ and $U = \{\spadesuit, \clubsuit, \heartsuit\}$.

$$Q \quad ; \quad R \quad = \quad Q\,;R$$

$$\{(1, \mathsf{b}), (2, \mathsf{b})\} \quad ; \quad \{(\mathsf{a}, \clubsuit), (\mathsf{b}, \clubsuit), (\mathsf{b}, \heartsuit)\} = \{(1, \clubsuit), (1, \heartsuit), (2, \clubsuit), (2, \heartsuit)\}$$



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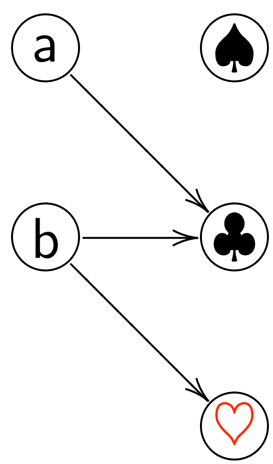

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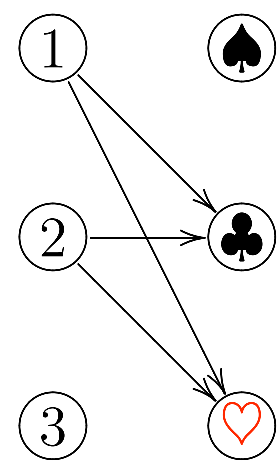

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad ; \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Composition of matrices is possible if sizes match.

Axiomatising heterogeneous relation algebra

Same axioms, but add typing and typing rules (the definition can also be based on category theory).

1. $R_{\mathsf{S},\mathsf{T}} : \mathsf{S} \leftrightarrow \mathsf{T}$ (relation $R_{\mathsf{S},\mathsf{T}}$ has type $\mathsf{S} \leftrightarrow \mathsf{T}$).
2. $Q_{\mathsf{S},\mathsf{T}} \sqcup R_{\mathsf{S},\mathsf{T}} : \mathsf{S} \leftrightarrow \mathsf{T}$.
3. $\overline{R_{\mathsf{S},\mathsf{T}}} : \mathsf{S} \leftrightarrow \mathsf{T}$.
4. $Q_{\mathsf{S},\mathsf{T}} \,;\, R_{\mathsf{T},\mathsf{U}} : \mathsf{S} \leftrightarrow \mathsf{U}$.
5. $R^{\smile}_{\mathsf{S},\mathsf{T}} : \mathsf{T} \leftrightarrow \mathsf{S}$.
6. $Q_{\mathsf{S},\mathsf{T}} \sqcup R_{\mathsf{U},\mathsf{V}}$ is defined only if $\mathsf{S} = \mathsf{U}$ and $\mathsf{T} = \mathsf{V}$.
7. $Q_{\mathsf{S},\mathsf{T}} \,;\, R_{\mathsf{U},\mathsf{V}}$ is defined only if $\mathsf{T} = \mathsf{U}$.
8. There are constants $\mathbb{I}_{\mathsf{S},\mathsf{S}}$, $\bot\!\!\!\bot_{\mathsf{S},\mathsf{T}}$, $\top\!\!\!\top_{\mathsf{S},\mathsf{T}}$ for each S and T.

Types are usually omitted from expressions. Thus, different instances of $\mathbb{I}$ may have different types (same remark for $\bot\!\!\!\bot$ and $\top\!\!\!\top$).

Laws that can be derived

Most laws derivable in the homogeneous setting are also derivable in the heterogeneous setting, but there are exceptions.

Example

- Homogeneous RA law: $\mathbb{T} ; \mathbb{T} = \mathbb{T}$.

  PROOF

  $$\mathbb{T} \;=\; \mathbb{I} ; \mathbb{T} \;\sqsubseteq\; \mathbb{T} ; \mathbb{T} \;\sqsubseteq\; \mathbb{T}$$

- Heterogeneous RA: $\mathbb{T}_{\mathsf{S,T}} ; \mathbb{T}_{\mathsf{T,U}} = \mathbb{T}_{\mathsf{S,U}}$ cannot be derived. The above proof cannot be used:

  $$\mathbb{T}_{\mathsf{S,U}} \;=\; \mathbb{I}_{\mathsf{S,S}} ; \mathbb{T}_{\mathsf{S,U}} \;\not\sqsubseteq\; \mathbb{T}_{\mathsf{S,T}} ; \mathbb{T}_{\mathsf{T,U}} \;\sqsubseteq\; \mathbb{T}_{\mathsf{S,U}}.$$

  And there is a counterexample. The only relation between sets S and $\emptyset$ is

  $$\mathbb{T}_{S,\emptyset} = \mathbb{L}_{S,\emptyset}$$

  Thus $\mathbb{T}_{S,\emptyset} ; \mathbb{T}_{\emptyset,S} = \mathbb{L}_{S,\emptyset} ; \mathbb{L}_{\emptyset,S} = \mathbb{L}_{S,S} \neq \mathbb{T}_{S,S}$ unless $S = \emptyset$.

Direct products (internal)

A pair of (**projection**) relations (π_1, π_2) is called a **direct product** iff

(a)	$\pi_1^\smile ; \pi_1 = \mathbb{I}$	π_1 functional and surjective
(b)	$\pi_2^\smile ; \pi_2 = \mathbb{I}$	π_2 functional and surjective
(c)	$\pi_1^\smile ; \pi_2 = \mathbb{T}$	any two elements can be paired
(d)	$\pi_1 ; \pi_1^\smile \sqcap \pi_2 ; \pi_2^\smile = \mathbb{I}$	π_1, π_2 total and construct all pairs in a unique way

Set model

$$\pi_1 \stackrel{\text{def}}{=} \{((s_1, s_2), s_1) \mid s_1 \in S_1 \ \wedge \ s_2 \in S_2\}$$

$$\pi_2 \stackrel{\text{def}}{=} \{((s_1, s_2), s_2) \mid s_1 \in S_1 \ \wedge \ s_2 \in S_2\}$$

1. $\pi_1^\smile ; \pi_1 \ = \ \{(s, s) \mid s \in S_1\} \ = \ I_{\mathsf{S_1},\mathsf{S_1}}$

2. $\pi_2^\smile ; \pi_2 \ = \ \{(s, s) \mid s \in S_2\} \ = \ I_{\mathsf{S_2},\mathsf{S_2}}$

3. $\pi_1^\smile ; \pi_2 \ = \ S_1 \times S_2 \ = \ V_{\mathsf{S_1},\mathsf{S_2}}$

4. $$\begin{aligned} \pi_1 ; \pi_1^\smile \cap \pi_2 ; \pi_2^\smile = \quad & \{((s_1, s_2), (s_1, s_2')) \mid s_1 \in S_1 \ \wedge \ s_2, s_2' \in S_2\} \\ & \cap \{((s_1, s_2), (s_1', s_2)) \mid s_1, s_1' \in S_1 \ \wedge \ s_2 \in S_2\} \\ = \quad & \{((s_1, s_2), (s_1, s_2')) \mid s_1, s_2 \in S_1 \ \wedge \ s_2' \in S_2\} \\ = \quad & I_{\mathsf{S_1 \times S_2},\mathsf{S_1 \times S_2}} \end{aligned}$$

Remark: Direct products for empty types

Consider the set model on the previous page. If $S_1 = \emptyset$ and $S_2 \neq \emptyset$, then both π_1 and π_2 are empty and π_2 cannot be surjective. If empty types have to be dealt with, only functionality of π_1 and π_2 should be required[1], i.e., $\pi_1^{\smile} ; \pi_1 \sqsubseteq \mathbb{I}$ and $\pi_2^{\smile} ; \pi_2 \sqsubseteq \mathbb{I}$.

[1] Thanks to Michael Winter for pointing that out to me.

Tupling and parallel product

$$\langle R_1, R_2] \stackrel{\text{def}}{=} R_1 ; \pi_1^{\smile} \ \sqcap \ R_2 ; \pi_2^{\smile}$$

$$[R_1, R_2] \stackrel{\text{def}}{=} \pi_1 ; R_1 ; \pi_1^{\smile} \ \sqcap \ \pi_2 ; R_2 ; \pi_2^{\smile}$$

Set model

$$\langle R_1, R_2] \ = \ \{(s, (s_1, s_2)) \mid sR_1s_1 \ \wedge \ sR_2s_2\}$$

$$[R_1, R_2] \ = \ \{((s_1, s_2), (s_1', s_2')) \mid s_1R_1s_1' \ \wedge \ s_2R_2s_2'\}$$

Compare with the cartesian product:

$$R_1 \times R_2 \ = \ \{((s_1, s_1'), (s_2, s_2')) \mid s_1R_1s_1' \ \wedge \ s_2R_2s_2'\}$$

Same cardinality, but different structure.

More general product

$$\pi_1 ; R_1 ; \rho_1^{\smile} \ \sqcap \ \pi_2 ; R_2 ; \rho_2^{\smile}$$

where (π_1, π_2) and (ρ_1, ρ_2) are direct products.

Matrix model (an example of parallel product)

The axioms: $\pi_1^{\smile} ; \pi_1 = \mathbb{I}$ $\quad \pi_2^{\smile} ; \pi_2 = \mathbb{I}$ $\quad \pi_1^{\smile} ; \pi_2 = \mathbb{T}$ $\quad \pi_1 ; \pi_1^{\smile} \sqcap \pi_2 ; \pi_2^{\smile} = \mathbb{I}$

$$\pi_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \pi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad R_2 = \begin{pmatrix} e & f & g \\ h & i & j \\ k & l & m \end{pmatrix}$$

$$a, \ldots, m \in \{0, 1\}$$

$$[R_1, R_2] = \left(\begin{array}{ccc|ccc} a \wedge e & a \wedge f & a \wedge g & b \wedge e & b \wedge f & b \wedge g \\ a \wedge h & a \wedge i & a \wedge j & b \wedge h & b \wedge i & b \wedge j \\ a \wedge k & a \wedge l & a \wedge m & b \wedge k & b \wedge l & b \wedge m \\ \hline c \wedge e & c \wedge f & c \wedge g & d \wedge e & d \wedge f & d \wedge g \\ c \wedge h & c \wedge i & c \wedge j & d \wedge h & d \wedge i & d \wedge j \\ c \wedge k & c \wedge l & c \wedge m & d \wedge k & d \wedge l & d \wedge m \end{array}\right)$$

Matrix model (an example of tupling)

The axioms: $\pi_1^{\smile} ; \pi_1 = \mathbb{I}$ $\quad \pi_2^{\smile} ; \pi_2 = \mathbb{I}$ $\quad \pi_1^{\smile} ; \pi_2 = \mathbb{T}$ $\quad \pi_1 ; \pi_1^{\smile} \sqcap \pi_2 ; \pi_2^{\smile} = \mathbb{I}$

$$\pi_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \pi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad R_2 = \begin{pmatrix} e & f & g \\ h & i & j \end{pmatrix}$$

$$a, \ldots, j \in \{0, 1\}$$

$$\langle R_1, R_2] = \left(\begin{array}{ccc|ccc} a \wedge e & a \wedge f & a \wedge g & b \wedge e & b \wedge f & b \wedge g \\ \hline c \wedge h & c \wedge i & c \wedge j & d \wedge h & d \wedge i & d \wedge j \end{array}\right)$$

Unsharpness

Let (π_1, π_2) be the direct product used in the tuplings and parallel products below. Let

$$[R_1, R_2\rangle \stackrel{\text{def}}{=} \pi_1 ; R_1 \;\sqcap\; \pi_2 ; R_2 \ .$$

The problem: [Cardoso 1982] Does

$$\langle Q_1, Q_2] ; [R_1, R_2\rangle = Q_1 ; R_1 \sqcap Q_2 ; R_2$$

hold for all relations Q_1, Q_2, R_1, R_2?

Unsharpness

Let (π_1, π_2) be the direct product used in the tuplings and parallel products below. Let

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The problem: [Cardoso 1982] Does

$$\langle Q_1, Q_2] ; [R_1, R_2\rangle = Q_1 ; R_1 \sqcap Q_2 ; R_2$$

hold for all relations Q_1, Q_2, R_1, R_2?

1. It holds for concrete algebras of relations and all representable RAs.
2. It holds in RA for many special cases [Zierer 88].
3. It does not hold in RA [Maddux 1993].
4. It holds in RA for the special cases

$$\langle Q_1, Q_2] ; [R_1, R_2] = \langle Q_1 ; R_1, Q_2 ; R_2]$$
$$[Q_1, Q_2] ; [R_1, R_2] = [Q_1 ; R_1, Q_2 ; R_2]$$

[Desharnais 1999]. The last equality was in fact the original problem of Cardoso and it was generalized to the one stated above.

Direct sums (internal)

A pair of (**injection**) relations (σ_1, σ_2) is called a **direct sum** iff

(a)	$\sigma_1 ; \sigma_1^{\smile} = \mathbb{I}$	σ_1 total and injective
(b)	$\sigma_2 ; \sigma_2^{\smile} = \mathbb{I}$	σ_2 total and injective
(c)	$\sigma_1 ; \sigma_2^{\smile} = \perp\!\!\!\perp$	σ_1 and σ_2 inject elements in disjoint subsets
(d)	$\sigma_1^{\smile} ; \sigma_1 \sqcup \sigma_2^{\smile} ; \sigma_2 = \mathbb{I}$	σ_1, σ_2 functional and construct all injected elements

Set model

$$\sigma_1 \stackrel{\text{def}}{=} \{(s, (s, 1)) \mid s \in S_1\}$$

$$\sigma_2 \stackrel{\text{def}}{=} \{(s, (s, 2)) \mid s \in S_2\}$$

1. $\sigma_1 ; \sigma_1^{\smile} \; = \; I_{\mathsf{S}_1, \mathsf{S}_1}$
2. $\sigma_2 ; \sigma_2^{\smile} \; = \; I_{\mathsf{S}_2, \mathsf{S}_2}$
3. $\sigma_1 ; \sigma_2^{\smile} \; = \; \emptyset_{\mathsf{S}_1, \mathsf{S}_2}$
4. $\sigma_1^{\smile} ; \sigma_1 \cap \sigma_2^{\smile} ; \sigma_2 = \{((s, 1), (s, 1)) \mid s \in S_1\} \cup \{((s, 2), (s, 2)) \mid s \in S_2\}$

   $= I_{\mathsf{S}_1 \uplus \mathsf{S}_2, \mathsf{S}_1 \uplus \mathsf{S}_2}$

Matrix model (an example of direct sum)

The axioms: $\sigma_1 ; \sigma_1^{\smile} = \mathbb{I}$ $\quad \sigma_2 ; \sigma_2^{\smile} = \mathbb{I}$ $\quad \sigma_1 ; \sigma_2^{\smile} = \perp\!\!\!\perp$ $\quad \sigma_1^{\smile} ; \sigma_1 \sqcup \sigma_2^{\smile} ; \sigma_2 = \mathbb{I}$

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad R_2 = \begin{pmatrix} e & f & g \\ h & i & j \\ k & l & m \end{pmatrix} \qquad a, \ldots, m \in \{0, 1\}$$

$$\sigma_1^{\smile} ; R_1 ; \sigma_1 \cup \sigma_2^{\smile} ; R_2 ; \sigma_2 = \left(\begin{array}{cc|ccc} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ \hline 0 & 0 & e & f & g \\ 0 & 0 & h & i & j \\ 0 & 0 & k & l & m \end{array}\right)$$

Expressivity

1. Maddux, page 33 in [Brink Kahl Schmidt 1997]:

   > An equation is true in every relation algebra iff its translation into a 3-variable sentence can be proved using at most 4 variables.

2. Maddux, page 36 in [Brink Kahl Schmidt 1997]:

   > Already in 1915 Löwenheim presented a proof (taken from a letter by Korselt) that the sentence saying "there are at least four elements", namely
   >
   > $$\exists v_0 \exists v_1 \exists v_2 \exists v_3 (\neg v_0 \mathbb{I} v_1 \wedge \neg v_0 \mathbb{I} v_2 \wedge \neg v_1 \mathbb{I} v_2 \wedge \neg v_0 \mathbb{I} v_3 \wedge \neg v_1 \mathbb{I} v_3 \wedge \neg v_2 \mathbb{I} v_3)$$
   >
   > is not equivalent to any relation-algebraic equation.

3. How to increase expressivity?

   - Add projections.
   - Use fork algebra: RA with an additional operator ∇ for pairing. See Haeberer *et al.*, Chapter 4 in [Brink Kahl Schmidt 1997].

Additional operators

1. Transitive closure *: add the axioms [Ng 1984]

   - $R \sqcup R\,;R^* = R\,;R^*$ (i.e., $R \sqsubseteq R\,;R^*$)
   - $(R\,;\overline{\overline{R}\,;R^\smile})^* = R\,;\overline{\overline{R}\,;R^\smile}$
   - $R^* \sqcup (R \sqcup Q)^* = (R \sqcup Q)^*$ (i.e., monotonicity $R^* \sqsubseteq (R \sqcup Q)^*$)

2. Left residual: largest solution X of $X\,;Q \sqsubseteq R$

   - Definition by a Galois connection: $X\,;Q \sqsubseteq R \;\Leftrightarrow\; X \sqsubseteq R/Q$
   - Explicit definition: $R/Q = \overline{\overline{R}\,;Q^\smile}$

3. Right residual: largest solution X of $Q\,;X \sqsubseteq R$

   - Definition by a Galois connection: $Q\,;X \sqsubseteq R \;\Leftrightarrow\; X \sqsubseteq Q\backslash R$
   - Explicit definition: $Q\backslash R = \overline{Q^\smile\,;\overline{R}}$

Complete relation algebras

1. A **complete** RA is an RA $\langle A, \sqcup, \overline{}, ;, \smile, \mathbb{I} \rangle$ for which

   $$\bigsqcup T \text{ exists for all } T \subseteq A$$

   (hence $\bigsqcap T$ exists too).

2. In a complete RA, monotonic functions have a least and greatest fixed point. For instance, R^* can be defined by

   $$R^* = (\mu X : \mathbb{I} \sqcup R ; X).$$

3. Useful, e.g., for program semantics.

4. Calculational rules for the manipulation of fixed points can be found in [Backhouse 2000].

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