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Demonic Algebra with Domain*

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Abstract. We first recall the concept of Kleene algebra with domain (KAD). Then we explain how to use the operators of KAD to define a demonic refinement ordering and demonic operators (many of these definitions come from the literature). Then, taking the properties of the KAD-based demonic operators as a guideline, we axiomatise an algebra that we call *Demonic algebra with domain* (DAD). The laws of DAD not concerning the domain operator agree with those given in the 1987 CACM paper *Laws of programming* by Hoare et al. Finally, we investigate the relationship between demonic algebras with domain and KAD-based demonic algebras. The question is whether every DAD is isomorphic to a KAD-based demonic algebra. We show that it is not the case in general. However, if a DAD $\mathcal{D}$ is isomorphic to a demonic algebra based on a KAD $\mathcal{K}$, then it is possible to construct a KAD isomorphic to $\mathcal{K}$ using the operators of $\mathcal{D}$. We also describe a few open problems.

1 Introduction

The basic operators of Kleene algebra (KA) or relation algebra (RA) can directly be used to give an abstract angelic semantics of while programs. For instance, $a + b$ corresponds to an angelic non-deterministic choice between programs a and b, and $(t \cdot b)^* \cdot \neg t$ is the angelic semantics of a loop with condition t and body b. One way to express demonic semantics in KA or RA is to define demonic operators in terms of the basic operators; these demonic operators can then be used in the semantic definitions. In RA, this has been done frequently (see for instance [1,2,6,7,18,22,26]); in KA, much less [12,13].

In the recent years, various algebras for program refinement have seen the day [3,14,15,16,24,25,27]. The refinement algebra of von Wright is an abstraction of predicate transformers, while the laws of programming of Hoare et al. have an underlying relational model. Möller's lazy Kleene algebra has weaker axioms than von Wright's and can handle systems in which infinite sequences of states may occur.

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Our goal is also to design a refinement algebra, that we call a *Demonic algebra* (DA). Rather than designing it with a concrete model in mind, our first goal is to come as close as possible to the kind of algebras that one gets by defining demonic operators in *KA with domain* (KAD) [9,10,11], as is done in [12,13], and then forgetting the basic angelic operators of KAD. Starting from KAD means that DA abstracts many concrete models, just like KA does. We hope that the closeness to KA will eventually lead to decision procedures like those of KA. A second longer term goal, not pursued here, is to precisely determine the relationship of DA with the other refinement algebras; we will say a few words about that in the conclusion.

In Section 2, we recall the definitions of Kleene algebra and its extensions, *Kleene algebra with tests* (KAT) and *Kleene algebra with domain* (KAD). This section also contains the definitions of demonic operators in terms of the KAD operators. Section 3 presents the axiomatisation of DA and its extensions, *DA with tests* (DAT) and *DA with domain* (DAD), as well as derived laws. It turns out that the laws of DAT closely correspond to the laws of programming of [14,15]. In Section 4, we begin to investigate the relationship between KAD and DAD by first defining angelic operators in terms of the demonic operators (call this transformation $\mathcal{G}$). Then we investigate whether the angelic operators thus defined by $\mathcal{G}$ induce a KAD. Not all answers are known there and we state a conjecture that we believe holds and from which the conditions that force $\mathcal{G}$ to induce a KAD can be determined. It is shown in Section 5 that the conjecture holds in those DADs obtained from a KAD by defining demonic operators in terms of the angelic operators (call this transformation $\mathcal{F}$). The good thing is that $\mathcal{F}$ followed by $\mathcal{G}$ is the identity. Section 6 simply states the main unsolved problem. We conclude in Section 7 with a description of future research.

2 Kleene Algebra with Domain and KAD-based Demonic Operators

In this section, we recall basic definitions about KA and its extensions, KAT and KAD. Then we present the KAD-based definition of the demonic operators.

Definition 1 (Kleene algebra). *A* Kleene algebra *(KA) [4,20] is a structure* $(K, +, \cdot, ^*, 0, 1)$ *such that the following properties hold for all* $x, y, z \in K$*.*

$$(x + y) + z = x + (y + z) \tag{1}$$
$$x + y = y + x \tag{2}$$
$$x + x = x \tag{3}$$
$$0 + x = x \tag{4}$$
$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \tag{5}$$
$$0 \cdot x = x \cdot 0 = 0 \tag{6}$$
$$1 \cdot x = x \cdot 1 = x \tag{7}$$
$$x \cdot (y + z) = x \cdot y + x \cdot z \tag{8}$$

$$(x + y) \cdot z = x \cdot z + y \cdot z \tag{9}$$

$$x^* = x^* \cdot x + 1 \tag{10}$$

Addition induces a partial order $\leq$ such that, for all $x, y \in K$,

$$x \leq y \iff x + y = y \ . \tag{11}$$

Finally, the following properties must be satisfied for all $x, y, z \in K$.

$$x \cdot z + y \leq z \implies x^* \cdot y \leq z \tag{12}$$

$$z \cdot x + y \leq z \implies y \cdot x^* \leq z \tag{13}$$

Remark 2. Hollenberg has shown that the following symmetric version of (10),

$$x^* = x \cdot x^* + 1 \ , \tag{14}$$

is derivable from these axioms [17] and Kozen has shown in [19] that (12) and (13) are independent.

One can show that $x^* = \mu_{\leq}(y :: y \cdot x + 1)$ with (7), (10) and (13) and that $x^* = \mu_{\leq}(y :: x \cdot y + 1)$ with (7), (14) and (12).

To reason about programs, it is useful to have a concept of condition, or test. It is provided by Kleene algebra with tests.

Definition 3 (Kleene algebra with tests). *A* KA with tests *(KAT) [21] is a structure $(K, \mathsf{test}(K), +, \cdot, ^*, 0, 1, \neg)$ such that $\mathsf{test}(K) \subseteq \{t \mid t \in K \wedge t \leq 1\}$, $(K, +, \cdot, ^*, 0, 1)$ is a KA and $(\mathsf{test}(K), +, \cdot, \neg, 0, 1)$ is a Boolean algebra.*

In the sequel, we use the letters s, t, u, v for tests and w, x, y, z for programs. The angelic semantics of programs is then given by the following, where $x [\!] y$ is the non-deterministic choice between x and y.

$$\begin{aligned} \text{abort} &= 0 \\ \text{skip} &= 1 \\ x [\!] y &= x + y \\ x; y &= x \cdot y \\ \text{if } t \text{ then } x \text{ else } y &= t \cdot x + \neg t \cdot y \\ \text{while } t \text{ do } x &= (t \cdot x)^* \cdot \neg t \end{aligned}$$

It is useful to have a grip on the inputs of the aforementioned programs. The notion of domain encapsulates the necessary properties.

Definition 4 (Kleene algebra with domain). *A* KA with domain *(KAD) [9,10,13,11] is a structure $(K, \mathsf{test}(K), +, \cdot, ^*, 0, 1, \neg, ^\ulcorner)$ such that $(K, \mathsf{test}(K), +, \cdot, ^*, 0, 1, \neg)$ is a KAT and, for all $x \in K$ and $t \in \mathsf{test}(K)$,*

$$x \leq {}^\ulcorner x \cdot x \ , \tag{15}$$

$${}^\ulcorner(t \cdot x) \leq t \ , \tag{16}$$

$${}^\ulcorner(x \cdot {}^\ulcorner y) \leq {}^\ulcorner(x \cdot y) \ . \tag{17}$$

It turns out that these axioms force the test algebra $\mathsf{test}(K)$ to be the maximal Boolean algebra included in $\{x \mid x \leq 1\}$ [11].

Example 5. This example illustrates the domain operator for the familiar model of relations.

$$\ulcorner\{(0,0),(0,1),(2,1)\} = \{(0,0),(2,2)\}$$
$$\ulcorner\{(0,0),(0,1),(0,2)\} = \{(0,0)\}$$
$$\ulcorner\{\} = \{\}$$

Note that (17) is satisfied in relational algebras. It is called *locality*. However, there are KATs where it is false; see [8] for a counter-example. There are many other properties about KAT and KAD and we gather those that will be used later on. See [11] or [13] for proofs.

Proposition 6. *The following hold for all $t \in \mathsf{test}(K)$ and all $x, y \in K$.*

1. $t \cdot t = t$
2. $t \cdot \neg t = \neg t \cdot t = 0$
3. $x = y \iff x \cdot t = y \cdot t \wedge x \cdot \neg t = y \cdot \neg t$
4. $\ulcorner x = \mathsf{min}_{\leq}\{t \mid t \in \mathsf{test}(K) \wedge t \cdot x = x\}$
5. $\ulcorner x \cdot x = x$
6. $\ulcorner x \leq t \iff x \leq t \cdot x$
7. $\ulcorner(x \cdot \ulcorner y) = \ulcorner(x \cdot y)$
8. $\neg\ulcorner x \cdot x = 0$
9. $\ulcorner t = t$
10. $\ulcorner(t \cdot x) = t \cdot \ulcorner x$
11. $\ulcorner(x + y) = \ulcorner x + \ulcorner y$
12. $x \leq y \Longrightarrow \ulcorner x \leq \ulcorner y$
13. $\ulcorner(x \cdot t) \leq t \iff \ulcorner(x^* \cdot t) \leq t$
14. $\ulcorner(x^*) = 1$

The following operator characterises the set of points from which no computation as described by x may lead outside the domain of y.

Definition 7 (KA-Implication). *Let x and y be two elements of a KAD. The* KA-implication $x \rightarrow y$ *is defined by* $x \rightarrow y = \neg\ulcorner(x \cdot \neg\ulcorner y)$.

We are now ready to introduce the demonic operators. Most proofs can be found in [13].

Definition 8 (Demonic refinement). *Let x and y be two elements of a KAD. We say that x* refines *y, noted $x \sqsubseteq_A y$, when $\ulcorner y \leq \ulcorner x$ and $\ulcorner y \cdot x \leq y$.*

The subscript A in $\sqsubseteq_A$ indicates that the demonic refinement is defined with the operators of the angelic world. An analogous notation will be introduced when we define angelic operators in the demonic world. It is easy to show that $\sqsubseteq_A$ is a partial order. Note that for all tests s and t, $s \sqsubseteq_A t \iff t \leq s$. This definition can be simply illustrated with relations. Let $Q = \{(1,2),(2,4)\}$ and $R = \{(1,2),(1,3)\}$. Then $\ulcorner R = \{(1,1)\} \subseteq \{(1,1),(2,2)\} = \ulcorner Q$. Since in addition $\ulcorner R; Q = \{(1,2)\} \subseteq R$, we have $Q \sqsubseteq_A R$ (";" is the usual relational composition).

Proposition 9 (Demonic upper semilattice).

1. *The partial order* $\sqsubseteq_A$ *induces an upper semilattice with* demonic join $\sqcup_A$*:*

$$x \sqsubseteq_A y \Longleftrightarrow x \sqcup_A y = y \ .$$

2. *Demonic join satisfies the following two properties.*

$$x \sqcup_A y = \ulcorner x \cdot \ulcorner y \cdot (x + y)$$
$$\ulcorner(x \sqcup_A y) = \ulcorner x \sqcup_A \ulcorner y = \ulcorner x \cdot \ulcorner y$$

Definition 10 (Demonic composition). *The* demonic composition *of two elements* x *and* y *of a KAD, written* $x \,\square_A\, y$*, is defined by* $x \,\square_A\, y = (x \rightarrow y) \cdot x \cdot y$*.*

Proposition 11. *Let* K *be a KAD with* $t \in \mathsf{test}(K)$ *and* $x, y, z \in K$*.*

1. $x \,\square_A\, (y \,\square_A\, z) = (x \,\square_A\, y) \,\square_A\, z$
2. $t \,\square_A\, x = t \cdot x$
3. *If* $\ulcorner y = 1$ *then* $x \,\square_A\, y = x \cdot y$
4. $\ulcorner(x \,\square_A\, y) = (x \rightarrow y) \cdot \ulcorner x$
5. $(x \rightarrow y) = (x \rightarrow \ulcorner y)$
6. $(x \rightarrow y) \cdot x = (x \rightarrow y) \cdot x \cdot \ulcorner y$
7. $(x + y) \rightarrow z = (x \rightarrow z) \cdot (y \rightarrow z)$
8. $(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z)$
9. $t \cdot ((t \cdot x) \rightarrow y) = t \cdot (x \rightarrow y)$
10. $\neg t \cdot ((t \cdot x) \rightarrow y) = \neg t$
11. $t \leq x \rightarrow t \iff t \leq x^* \rightarrow t$
12. $x \leq y \implies y \rightarrow z \leq x \rightarrow z$
13. $x \square y \leq x \cdot y$
14. $x \sqsubseteq_A y \implies x \square z \sqsubseteq_A y \square z$
15. $x \sqsubseteq_A y \implies z \square x \sqsubseteq_A z \square y$

Definition 12 (Demonic star). *Let* $x \in K$*, where* K *is a KAD. The unary iteration operator* ${}^{\times_A}$ *is defined by* $x^{\times_A} = x^* \,\square_A\, \ulcorner x$*.*

Proposition 13. *Let* $x, y, z \in K$*, where* K *is a KAD.*

1. $x^{\times_A} = x^{\times_A} \,\square_A\, x \sqcup_A 1$
2. $x \,\square_A\, z \sqsubseteq_A z \implies x^{\times_A} \,\square_A\, z \sqsubseteq_A z$
3. $z \,\square_A\, x \sqsubseteq_A z \implies z \,\square_A\, x^{\times_A} \sqsubseteq_A z$
4. $x \,\square_A\, z \sqcup_A y \sqsubseteq_A z \implies x^{\times_A} \,\square_A\, y \sqsubseteq_A z$
5. $z \,\square_A\, x \sqcup_A y \sqsubseteq_A z \implies y \,\square_A\, x^{\times_A} \sqsubseteq_A z$

Proof.

1. $\quad x^{\times_A} \,\square_A\, x \sqcup_A 1$

$=$ ⟨ by Definition 12 and Proposition11-1 ⟩

$\quad x^* \,\square_A\, (\ulcorner x \,\square_A\, x) \sqcup_A 1$

$=$ $\quad \langle$ by Proposition 11-2 and Proposition 6-5 $\rangle$

$\quad x^* \square_A x \sqcup_A 1$

$=$ $\quad \langle$ by Proposition 9, Proposition 6-9 and (7) $\rangle$

$\quad \ulcorner(x^* \square_A x) \cdot (x^* \square_A x + 1)$

$=$ $\quad \langle$ by Proposition 11-4 and Definition 10 $\rangle$

$\quad (x^* \rightarrow x) \cdot \ulcorner(x^*) \cdot ((x^* \rightarrow x) \cdot x^* \cdot x + 1)$

$=$ $\quad \langle$ because $\ulcorner(x^*) = 1$ and by (7) $\rangle$

$\quad (x^* \rightarrow x) \cdot ((x^* \rightarrow x) \cdot x^* \cdot x + 1)$

$=$ $\quad \langle$ by (8) and Proposition 6-1 $\rangle$

$\quad (x^* \rightarrow x) \cdot (x^* \cdot x + 1)$

$=$ $\quad \langle$ by (10) $\rangle$

$\quad (x^* \rightarrow x) \cdot x^*$

$=$ $\quad \langle$ by Proposition 11-6 and Proposition 6-5 $\rangle$

$\quad (x^* \rightarrow x) \cdot x^* \cdot \ulcorner x$

$=$ $\quad \langle$ by Definition 10 and Definition 12 $\rangle$

$\quad x^{\times_A}$

2. $\quad x^{\times_A} \square_A z \sqsubseteq_A z$

$\Longleftrightarrow$ $\quad \langle$ by Definition 12 and Proposition11-1 $\rangle$

$\quad x^* \square (\ulcorner x \square_A z) \sqsubseteq_A z$

$\Longleftrightarrow$ $\quad \langle$ by Proposition 11-2 $\rangle$

$\quad x^* \square_A (\ulcorner x \cdot z) \sqsubseteq_A z$

$\Longleftrightarrow$ $\quad \langle$ by Definition 8 $\rangle$

$\quad \ulcorner z \leq \ulcorner(x^* \square_A (\ulcorner x \cdot z)) \ \wedge \ \ulcorner z \cdot (x^* \square_A (\ulcorner x \cdot z)) \leq z$

$\Longleftrightarrow$ $\quad \langle$ by Proposition 11-4 and Definition 10 $\rangle$

$\quad \ulcorner z \leq (x^* \rightarrow (\ulcorner x \cdot z)) \cdot \ulcorner(x^*) \ \wedge \ \ulcorner z \cdot (x^* \rightarrow (\ulcorner x \cdot z)) \cdot x^* \cdot \ulcorner x \cdot z \leq z$

$\Longleftrightarrow$ $\quad \langle$ by Proposition 6-14 and (7) $\rangle$

$\quad \ulcorner z \leq x^* \rightarrow (\ulcorner x \cdot z) \ \wedge \ \ulcorner z \cdot (x^* \rightarrow (\ulcorner x \cdot z)) \cdot x^* \cdot \ulcorner x \cdot z \leq z$

$\Longleftrightarrow$ $\quad \langle$ predicate logic and Boolean algebra $\rangle$

$\quad \ulcorner z \leq x^* \rightarrow (\ulcorner x \cdot z) \ \wedge \ \ulcorner z \cdot x^* \cdot \ulcorner x \cdot z \leq z$

$\Longleftarrow$ $\quad \langle$ by Proposition 6-5 and Boolean algebra, $\ulcorner z \leq \ulcorner x$ implies
$z = \ulcorner z \cdot z = \ulcorner x \cdot \ulcorner z \cdot z = \ulcorner x \cdot z$ $\rangle$

$\quad \ulcorner z \leq \ulcorner x \ \wedge \ \ulcorner z \leq x^* \rightarrow z \ \wedge \ \ulcorner z \cdot x^* \cdot z \leq z$

$\Longleftrightarrow$ $\quad \langle$ by Proposition 11-5, (9) and Boolean algebra $\rangle$

$\quad \ulcorner z \leq \ulcorner x \ \wedge \ \ulcorner z \leq x^* \rightarrow \ulcorner z \ \wedge \ \ulcorner z \cdot (\ulcorner z \cdot x + \neg \ulcorner z \cdot x)^* \cdot z \leq z$

$\Longleftrightarrow$ $\quad \langle$ by Proposition 11-11 and the law $(x+y)^* = (x^* \cdot y)^* \cdot x^*$ $\rangle$

$\quad \ulcorner z \leq \ulcorner x \ \wedge \ \ulcorner z \leq x \rightarrow \ulcorner z \ \wedge \ \ulcorner z \cdot ((\ulcorner z \cdot x)^* \cdot \neg \ulcorner z \cdot x)^* \cdot (\ulcorner z \cdot x)^* \cdot z \leq z$

$\iff$ $\langle$ from $\ulcorner z \leq x \rightarrow \ulcorner z$, Boolean algebra, and Propositions 11-4 and 11-6, we get $\ulcorner z \cdot x = \ulcorner z \cdot (x \rightarrow \ulcorner z) \cdot x = \ulcorner z \cdot (x \rightarrow \ulcorner z) \cdot x \cdot \ulcorner z = \ulcorner z \cdot x \cdot \ulcorner z$ $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow \ulcorner z \;\wedge\; \ulcorner z \cdot ((\ulcorner z \cdot x \cdot \ulcorner z)^* \cdot \neg\ulcorner z \cdot x)^* \cdot (\ulcorner z \cdot x)^* \cdot z \leq z$$

$\iff$ $\langle$ by (10) $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow \ulcorner z \;\wedge\;$$
$$\ulcorner z \cdot \Big(\big((\ulcorner z \cdot x \cdot \ulcorner z)^* \cdot \ulcorner z \cdot x \cdot \ulcorner z + 1\big) \cdot \neg\ulcorner z \cdot x\Big)^* \cdot (\ulcorner z \cdot x)^* \cdot z \leq z$$

$\iff$ $\langle$ by (9), (4) and because $\ulcorner z \cdot \neg\ulcorner z = 0$ $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow \ulcorner z \;\wedge\; \ulcorner z \cdot (\neg\ulcorner z \cdot x)^* \cdot (\ulcorner z \cdot x)^* \cdot z \leq z$$

$\iff$ $\langle$ by Proposition 11-5 and (14) $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow z \;\wedge\; \ulcorner z \cdot (\neg\ulcorner z \cdot x \cdot (\neg\ulcorner z \cdot x)^* + 1) \cdot (\ulcorner z \cdot x)^* \cdot z \leq z$$

$\iff$ $\langle$ by (8), (4), (7) and because $\ulcorner z \cdot \neg\ulcorner z = 0$ $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow z \;\wedge\; \ulcorner z \cdot (\ulcorner z \cdot x)^* \cdot z \leq z$$

$\iff$ $\langle$ law of domain: $y \leq z \iff \ulcorner z \cdot y \leq z$ $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow z \;\wedge\; (\ulcorner z \cdot x)^* \cdot z \leq z$$

$\Longleftarrow$ $\langle$ by (12) with $x, y := \ulcorner z \cdot x, z$ $\rangle$

$$\ulcorner z \leq \ulcorner x \;\wedge\; \ulcorner z \leq x \rightarrow z \;\wedge\; \ulcorner z \cdot x \cdot z \leq z$$

$\iff$ $\langle$ by Boolean algebra and predicate logic $\rangle$

$$\ulcorner z \leq (x \rightarrow z) \cdot \ulcorner x \;\wedge\; \ulcorner z \cdot (x \rightarrow z) \cdot x \cdot z \leq z$$

$\iff$ $\langle$ by Proposition 11-4 and Definition 10 $\rangle$

$$\ulcorner z \leq \ulcorner(x \,\square_A\, z) \;\wedge\; \ulcorner z \cdot (x \,\square_A\, z) \leq z$$

$\iff$ $\langle$ by Definition 8 $\rangle$

$$x \,\square_A\, z \sqsubseteq_A z$$

3. $z \,\square_A\, x \sqsubseteq_A z$

$\iff$ $\langle$ by Definition 8 $\rangle$

$$\ulcorner z \leq \ulcorner(z \,\square_A\, x) \;\wedge\; \ulcorner z \cdot (z \,\square_A\, x) \leq z$$

$\iff$ $\langle$ by Proposition 11-4 and Definition 10 $\rangle$

$$\ulcorner z \leq (z \rightarrow x) \cdot \ulcorner z \;\wedge\; \ulcorner z \cdot (z \rightarrow x) \cdot z \cdot x \leq z$$

$\iff$ $\langle$ by Boolean algebra and Proposition 6-5 $\rangle$

$$\ulcorner z \leq (z \rightarrow x) \cdot \ulcorner z \;\wedge\; z \cdot x \leq z$$

$\Longrightarrow$ $\langle$ by Proposition 11-5, and (13) with $y := z$ $\rangle$

$$\ulcorner z \leq (z \rightarrow \ulcorner x) \cdot \ulcorner z \;\wedge\; z \cdot x^* \leq z$$

This derivation thus gives

$$\ulcorner z \leq (z \rightarrow \ulcorner x) \cdot \ulcorner z \ , \qquad (18)$$

$$z \cdot x^* \leq z \ . \qquad (19)$$

$$\ulcorner z$$
$$\leq \quad \langle \text{ by (18) } \rangle$$
$$(z \to \ulcorner x) \cdot \ulcorner z$$
$$\leq \quad \langle \text{ by (19) and Proposition 11-12 } \rangle$$
$$((z \cdot x^*) \to \ulcorner x) \cdot \ulcorner z$$
$$= \quad \langle \text{ by Proposition 11-8 } \rangle$$
$$(z \to (x^* \to \ulcorner x)) \cdot \ulcorner z$$
$$= \quad \langle \text{ by Proposition6-14 and (7) } \rangle$$
$$(z \to ((x^* \to \ulcorner x) \cdot \ulcorner(x^*))) \cdot \ulcorner z$$
$$= \quad \langle \text{ by Propositions 11-4 and 11-5 } \rangle$$
$$(z \to (x^* \square_A \ulcorner x)) \cdot \ulcorner z$$
$$= \quad \langle \text{ by Proposition 11-4 } \rangle$$
$$\ulcorner(z \square_A (x^* \square_A \ulcorner x))$$
$$= \quad \langle \text{ by Definition 12 } \rangle$$
$$\ulcorner(z \square_A x^{\times_A})$$

And the last inequality goes like this.

$$\ulcorner z \cdot (z \square_A x^{\times_A})$$
$$= \quad \langle \text{ by Definition 12 } \rangle$$
$$\ulcorner z \cdot (z \square_A (x^* \square_A \ulcorner x))$$
$$\leq \quad \langle \text{ Proposition 10-13 } \rangle$$
$$\ulcorner z \cdot z \cdot (x^* \square_A \ulcorner x)$$
$$\leq \quad \langle \text{ Proposition 10-13 } \rangle$$
$$\ulcorner z \cdot z \cdot x^* \cdot \ulcorner x$$
$$= \quad \langle \text{ by (19) and because } \ulcorner z \leq 1 \text{ and } \ulcorner x \leq 1 \ \rangle$$
$$z$$

The result then follows from Definition 8.

4. Suppose $x \square_A z \sqcup_A y \sqsubseteq_A z$. Then $y \sqsubseteq_A z$ and $x \square_A z \sqsubseteq_A z$ by Proposition 9. Then Part 2 of the present proposition gives $x^{\times_A} \square_A z \sqsubseteq_A z$.

$$x^{\times_A} \square_A y$$
$$\sqsubseteq_A \quad \langle \text{ since } y \sqsubseteq_A z \text{ and by Proposition 11-15 } \rangle$$
$$x^{\times_A} \square_A z$$
$$\sqsubseteq_A \quad \langle \text{ since } x^{\times_A} \square_A z \sqsubseteq_A z \ \rangle$$
$$z$$

5. The proof is similar to the previous one. □

Definition 14 (Conditional). *For each $t \in \mathsf{test}(K)$ and $x, y \in K$, the t-conditional is defined by $x \sqcap_{A t}\, y = t \cdot x + \neg t \cdot y$. The family of t-conditionals corresponds to a single ternary operator $\sqcap_{A \bullet}$ taking as arguments a test t and two arbitrary elements x and y.*

The demonic join operator $\sqcup_A$ is used to give the semantics of demonic nondeterministic choices and $\Box_A$ is used for sequences. Among the interesting properties of $\Box_A$, we cite $t \Box_A x = t \cdot x$ (Proposition 11-2), which says that composing a test t with an arbitrary element x is the same in the angelic and demonic worlds, and $x \Box_A y = x \cdot y$ if $\ulcorner y = 1$ (Proposition 11-3), which says that if the second element of a composition is total, then again the angelic and demonic compositions coincide. The ternary operator $\sqcap_{A \bullet}$ is similar to the conditional choice operator $\_ \lhd \_ \rhd \_$ of Hoare et al. [14,15]. It corresponds to a guarded choice with disjoint alternatives. The iteration operator $^{\times_A}$ rejects the finite computations that go through a state from which it is possible to reach a state where no computation is defined (e.g., due to blocking or abnormal termination).

We now present three theorems about the demonic operators introduced in this section, Theorems 15, 16 and 17. They consist of laws that will be taken as axioms of demonic algebra with domain in Section 3. Theorem 15 contains laws relating $\sqcup_A$, $\Box_A$ and $^{\times_A}$. Theorem 16 concerns the t-conditional $\sqcap_{A t}$. And Theorem 17 is about the relationship between $\sqcup_A$, $\Box_A$ and $\ulcorner$.

As usual, unary operators have the highest precedence, and demonic composition $\Box_A$ binds stronger than $\sqcup_A$ and $\sqcap_{A \bullet}$, which have the same precedence.

Theorem 15. *Let K be a KAD. The following properties hold for all $x, y, z \in K$.*

1. $x \sqcup_A (y \sqcup_A z) = (x \sqcup_A y) \sqcup_A z$
2. $x \sqcup_A y = y \sqcup_A x$
3. $x \sqcup_A x = x$
4. $0 \sqcup_A x = 0$
5. $x \Box_A (y \Box_A z) = (x \Box_A y) \Box_A z$
6. $0 \Box_A x = x \Box_A 0 = 0$
7. $1 \Box_A x = x \Box_A 1 = x$
8. $x \Box_A (y \sqcup_A z) = x \Box_A y \sqcup_A x \Box_A z$
9. $(x \sqcup_A y) \Box_A z = x \Box_A z \sqcup_A y \Box_A z$
10. $x^{\times_A} = x^{\times_A} \Box_A x \sqcup_A 1$
11. $x \sqsubseteq_A y \iff x \sqcup_A y = y$
12. $z \Box_A x \sqcup_A y \sqsubseteq_A z \implies y \Box_A x^{\times_A} \sqsubseteq_A z$
13. $x \Box_A z \sqcup_A y \sqsubseteq_A z \implies x^{\times_A} \Box_A y \sqsubseteq_A z$

Proof. See [13] for the proof of 1 to 9 and 11. Refer to Proposition 13 for the proof of 10, 12 and 13. □

Theorem 16. *Let K be a KAD. The following properties hold for all $s, t, u \in$ $\mathsf{test}(K)$ and all $x, y, z \in K$.*

1. $1 \sqsubseteq_A s$

2. $s \sqcap_{A\,t} u \in \mathsf{test}(K)$
3. $\neg t = 0 \sqcap_{A\,t} 1$
4. $x \sqcap_{A\,t} y = y \sqcap_{A\,\neg t} x$
5. $(t \mathbin{\square_A} x) \sqcap_{A\,t} y = x \sqcap_{A\,t} y$
6. $x \sqcap_{A\,t} x = x$
7. $x \sqcap_{A\,t} 0 = t \mathbin{\square_A} x$
8. $(x \sqcap_{A\,t} y) \mathbin{\square_A} z = (x \mathbin{\square_A} z) \sqcap_{A\,t} (y \mathbin{\square_A} z)$
9. $s \mathbin{\square_A} (x \sqcap_{A\,t} y) = (s \mathbin{\square_A} x) \sqcap_{A\,t} (s \mathbin{\square_A} y)$
10. $x \sqcap_{A\,t} (y \sqcup_A z) = (x \sqcap_{A\,t} y) \sqcup_A (x \sqcap_{A\,t} z)$
11. $x \sqcup_A (y \sqcap_{A\,t} z) = (x \sqcup_A y) \sqcap_{A\,t} (x \sqcup_A z)$
12. $t \sqcup_A \neg t = 0$
13. $\neg(1 \sqcap_{A\,t} s) = \neg t \sqcup_A \neg s$

Proof.

1. By Boolean algebra and Proposition 6-9, $\ulcorner s \le 1$ and $\ulcorner s \cdot 1 = \ulcorner s = s$, so $1 \sqsubseteq_A s$.
2. $s \sqcap_{A\,t} u$
   $=$ ⟨ by Definition 14 ⟩
   $t \cdot s + \neg t \cdot u$
   $\in$ ⟨ by Boolean algebra and definition of $\mathsf{test}(K)$ ⟩
   $\mathsf{test}(K)$
3. $0 \sqcap_{A\,t} 1$
   $=$ ⟨ by Definition 14 ⟩
   $t \cdot 0 + \neg t \cdot 1$
   $=$ ⟨ by Boolean algebra ⟩
   $\neg t$
4. $x \sqcap_{A\,t} y$
   $=$ ⟨ by Definition 14 ⟩
   $t \cdot x + \neg t \cdot y$
   $=$ ⟨ by (2) and Boolean algebra ⟩
   $\neg t \cdot y + \neg(\neg t) \cdot x$
   $=$ ⟨ by Definition 14 ⟩
   $y \sqcap_{A\,\neg t} x$
5. $(t \mathbin{\square_A} x) \sqcap_{A\,t} y$
   $=$ ⟨ by Definition 14 and Proposition 11-2 ⟩
   $t \cdot t \cdot x + \neg t \cdot y$
   $=$ ⟨ by Boolean algebra ⟩
   $t \cdot x + \neg t \cdot y$
   $=$ ⟨ by Definition 14 ⟩
   $x \sqcap_{A\,t} y$

6. $x \mathbin{\overline{\sqcap}_{A\,t}} x$

$=$ $\langle$ by Definition 14 $\rangle$

$t \cdot x + \neg t \cdot x$

$=$ $\langle$ by (9) $\rangle$

$(t + \neg t) \cdot x$

$=$ $\langle$ by Boolean algebra and (7) $\rangle$

x

7. $x \mathbin{\overline{\sqcap}_{A\,t}} 0$

$=$ $\langle$ by Definition 14 $\rangle$

$t \cdot x + \neg t \cdot 0$

$=$ $\langle$ by (6) and (4) $\rangle$

$t \cdot x$

$=$ $\langle$ by Proposition 11-2 $\rangle$

$t \mathbin{\square_A} x$

8. $(x \mathbin{\overline{\sqcap}_{A\,t}} y) \mathbin{\square_A} z$

$=$ $\langle$ by Definition 14 $\rangle$

$(t \cdot x + \neg t \cdot y) \mathbin{\square_A} z$

$=$ $\langle$ by Definition 10 $\rangle$

$((t \cdot x + \neg t \cdot y) \to z) \cdot (t \cdot x + \neg t \cdot y) \cdot z$

$=$ $\langle$ by Proposition 11-7 $\rangle$

$((t \cdot x) \to z) \cdot ((\neg t \cdot y) \to z) \cdot (t \cdot x + \neg t \cdot y) \cdot z$

$=$ $\langle$ by (8) $\rangle$

$(((t \cdot x) \to z) \cdot ((\neg t \cdot y) \to z) \cdot t \cdot x +$
$((t \cdot x) \to z) \cdot ((\neg t \cdot y) \to z) \cdot \neg t \cdot y) \cdot z$

$=$ $\langle$ by Proposition 11-9 and Boolean algebra $\rangle$

$(((\neg t \cdot y) \to z) \cdot t \cdot (x \to z) \cdot x + ((t \cdot x) \to z) \cdot \neg t \cdot (y \to z) \cdot y) \cdot z$

$=$ $\langle$ by Proposition 11-10 $\rangle$

$(t \cdot (x \to z) \cdot x + \neg t \cdot (y \to z) \cdot y) \cdot z$

$=$ $\langle$ by (9) $\rangle$

$t \cdot (x \to z) \cdot x \cdot z + \neg t \cdot (y \to z) \cdot y \cdot z$

$=$ $\langle$ by Definition 10 $\rangle$

$t \cdot (x \mathbin{\square_A} z) + \neg t \cdot (y \mathbin{\square_A} z)$

$=$ $\langle$ by Definition 14 $\rangle$

$(x \mathbin{\square_A} z) \mathbin{\overline{\sqcap}_{A\,t}} (y \mathbin{\square_A} z)$

9. $s \mathbin{\square_A} (x \mathbin{\overline{\sqcap}_{A\,t}} y)$

$=$ $\langle$ by Definition 14 and Proposition 11-2 $\rangle$

$$s \cdot (t \cdot x + \neg t \cdot y)$$
$= \quad \langle$ by Boolean algebra and (8) $\rangle$
$$t \cdot s \cdot x + \neg t \cdot s \cdot y$$
$= \quad \langle$ by Definition 14 and Proposition 11-2 $\rangle$
$$(s \square_A x) \sqcap_{A_t} (s \square_A y)$$

10. $$(x \sqcap_{A_t} y) \sqcup_A (x \sqcap_{A_t} z)$$
$= \quad \langle$ by Definition 14 $\rangle$
$$(t \cdot x + \neg t \cdot y) \sqcup_A (t \cdot x + \neg t \cdot z)$$
$= \quad \langle$ by Proposition 6-11, Proposition 6-10 and Proposition 9 $\rangle$
$$(t \cdot \ulcorner x + \neg t \cdot \ulcorner y) \cdot (t \cdot \ulcorner x + \neg t \cdot \ulcorner z) \cdot (t \cdot x + \neg t \cdot y + t \cdot x + \neg t \cdot z)$$
$= \quad \langle$ by (2), (3) and (8) $\rangle$
$$(t \cdot \ulcorner x + \neg t \cdot \ulcorner y) \cdot (t \cdot \ulcorner x + \neg t \cdot \ulcorner z) \cdot (t \cdot x + \neg t \cdot (y + z))$$
$= \quad \langle$ by Boolean algebra $\rangle$
$$(t \cdot \ulcorner x + \neg t \cdot \ulcorner y \cdot \ulcorner z) \cdot (t \cdot x + \neg t \cdot (y + z))$$
$= \quad \langle$ by (8), (9), Boolean algebra and Proposition 6-5 $\rangle$
$$t \cdot x + \neg t \cdot \ulcorner y \cdot \ulcorner z \cdot (y + z)$$
$= \quad \langle$ by Proposition 9 $\rangle$
$$t \cdot x + \neg t \cdot (y \sqcup_A z)$$
$= \quad \langle$ by Definition 14 $\rangle$
$$x \sqcap_{A_t} (y \sqcup_A z)$$

11. $$(x \sqcup_A y) \sqcap_{A_t} (x \sqcup_A z)$$
$= \quad \langle$ by Definition 14 $\rangle$
$$t \cdot (x \sqcup_A y) + \neg t \cdot (x \sqcup_A z)$$
$= \quad \langle$ by Proposition 9 $\rangle$
$$t \cdot \ulcorner x \cdot \ulcorner y \cdot (x + y) + \neg t \cdot \ulcorner x \cdot \ulcorner z \cdot (x + z)$$
$= \quad \langle$ by (8), (9), Boolean algebra and Proposition 6-5 $\rangle$
$$\ulcorner x \cdot (t \cdot \ulcorner y + \neg t \cdot \ulcorner z) \cdot (x + (t \cdot y + \neg t \cdot z))$$
$= \quad \langle$ by Proposition 6-11, Proposition 6-10 and Proposition 9 $\rangle$
$$x \sqcup_A (t \cdot y + \neg t \cdot z)$$
$= \quad \langle$ by Definition 14 $\rangle$
$$x \sqcup_A (y \sqcap_{A_t} z)$$

12. $$t \sqcup_A \neg t$$
$= \quad \langle$ by Proposition 6-9 and Proposition 9 $\rangle$
$$t \cdot \neg t \cdot (t + \neg t)$$
$= \quad \langle$ by Boolean algebra $\rangle$

$$
\begin{aligned}
& 0 \\
\text{13.} \quad & \neg(1 \mathbin{\bar{\sqcap}}_{A\,t}\, s) \\
= \quad & \qquad \langle \text{ by Definition 14 } \rangle \\
& \neg(t \cdot 1 + \neg t \cdot s) \\
= \quad & \qquad \langle \text{ by Boolean algebra } \rangle \\
& \neg t \cdot \neg s \cdot (\neg t + \neg s) \\
= \quad & \qquad \langle \text{ by Proposition 6-9 and Proposition 9 } \rangle \\
& \neg t \sqcup_A \neg s
\end{aligned}
$$

□

Theorem 17. *Let K be a KAD. The following properties hold for all $t \in \mathsf{test}(K)$ and all $x, y \in K$.*

1. $\ulcorner(x \mathbin{\square_A} t) \mathbin{\square_A} x = x \mathbin{\square_A} t$
2. $\ulcorner(x \mathbin{\square_A} y) = \ulcorner(x \mathbin{\square_A} \ulcorner y)$
3. $\ulcorner(x \sqcup_A y) = \ulcorner x \sqcup_A \ulcorner y$

Proof.

$$
\begin{aligned}
\text{1.} \quad & \ulcorner(x \mathbin{\square_A} t) \mathbin{\square_A} x \\
= \quad & \qquad \langle \text{ by Propositions 11-2 and 11-4 } \rangle \\
& (x \to t) \cdot x \\
= \quad & \qquad \langle \text{ by Propositions 11-6 and 6-9 } \rangle \\
& (x \to t) \cdot x \cdot t \\
= \quad & \qquad \langle \text{ by Definition 10 } \rangle \\
& x \mathbin{\square_A} t \\
\text{2.} \quad & \ulcorner(x \mathbin{\square_A} y) \\
= \quad & \qquad \langle \text{ by Proposition 11-4 } \rangle \\
& (x \to y) \cdot \ulcorner x \\
= \quad & \qquad \langle \text{ by Proposition 11-5 } \rangle \\
& (x \to \ulcorner y) \cdot \ulcorner x \\
= \quad & \qquad \langle \text{ by Proposition 11-4 } \rangle \\
& \ulcorner(x \mathbin{\square_A} \ulcorner y) \\
\text{3.} \quad & \ulcorner(x \sqcup_A y) \\
= \quad & \qquad \langle \text{ by Proposition 9 } \rangle \\
& \ulcorner x \cdot \ulcorner y \\
= \quad & \qquad \langle \text{ by Boolean algebra } \rangle \\
& \ulcorner x \cdot \ulcorner y \cdot (\ulcorner x + \ulcorner y) \\
= \quad & \qquad \langle \text{ by Proposition 9 } \rangle \\
& \ulcorner x \sqcup_A \ulcorner y
\end{aligned}
$$

□

3 Axiomatisation of Demonic Algebra with Domain

The demonic operators introduced at the end of the last section satisfy many properties. We choose some of them —more precisely, those of Theorems 15, 16 and 17— to become axioms of a new structure called demonic algebra with domain. For this definition, we follow the same path as for the definition of KAD. That is, we first define demonic algebra, then demonic algebra with tests and, finally, demonic algebra with domain.

3.1 Demonic Algebra

Demonic algebra, like KA, has a sum, a composition and an iteration operator. Here is its definition.

Definition 18 (Demonic algebra). *A* demonic algebra *(DA) is a structure* $(A_{\mathcal{D}}, \sqcup, \Box, ^{\times}, 0, 1)$ *such that the following properties are satisfied for* $x, y, z \in A_{\mathcal{D}}$.

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \tag{20}$$

$$x \sqcup y = y \sqcup x \tag{21}$$

$$x \sqcup x = x \tag{22}$$

$$0 \sqcup x = 0 \tag{23}$$

$$x \Box (y \Box z) = (x \Box y) \Box z \tag{24}$$

$$0 \Box x = x \Box 0 = 0 \tag{25}$$

$$1 \Box x = x \Box 1 = x \tag{26}$$

$$x \Box (y \sqcup z) = x \Box y \sqcup x \Box z \tag{27}$$

$$(x \sqcup y) \Box z = x \Box z \sqcup y \Box z \tag{28}$$

$$x^{\times} = x^{\times} \Box x \sqcup 1 \tag{29}$$

There is a partial order $\sqsubseteq$ *induced by* $\sqcup$ *such that for all* $x, y \in A_{\mathcal{D}}$,

$$x \sqsubseteq y \iff x \sqcup y = y \ . \tag{30}$$

The next two properties are also satisfied for all $x, y, z \in A_{\mathcal{D}}$.

$$x \Box z \sqcup y \sqsubseteq z \implies x^{\times} \Box y \sqsubseteq z \tag{31}$$

$$z \Box x \sqcup y \sqsubseteq z \implies y \Box x^{\times} \sqsubseteq z \tag{32}$$

When comparing Definitions 1 and 18, one observes the obvious correspondences $+ \leftrightarrow \sqcup, \cdot \leftrightarrow \Box, ^* \leftrightarrow ^{\times}, 0 \leftrightarrow 0, 1 \leftrightarrow 1$. The only difference in the axiomatisation between KA and DA is that 0 is the left and right identity of addition in KA (+), while it is a left and right zero of addition in DA ($\sqcup$). However, this minor difference has a rather important impact. While KAs and DAs are upper semilattices with + as the join operator for KAs and $\sqcup$ for DAs, the element 0 is the bottom of the semilattice for KAs and the top of the semilattice for DAs. Indeed, by (23) and (30),

$$x \sqsubseteq 0 \tag{33}$$

for all $x \in A_{\mathcal{D}}$.

All operators are monotonic with respect to the refinement ordering $\sqsubseteq$. That is, for all $x, y, z \in A_{\mathcal{D}}$,

$$x \sqsubseteq y \implies z \sqcup x \sqsubseteq z \sqcup y \ \wedge\ z \mathbin{\square} x \sqsubseteq z \mathbin{\square} y \ \wedge\ x \mathbin{\square} z \sqsubseteq y \mathbin{\square} z \ \wedge\ x^{\times} \sqsubseteq y^{\times} \ .$$

Monotonicity of $\sqcup$ and $\square$ can easily be derived from (30), (27) and (28). That of $^{\times}$ is shown from (29) and (32) as follows:

$$x \sqsubseteq y \implies y^{\times} \mathbin{\square} x \sqcup 1 \sqsubseteq y^{\times} \mathbin{\square} y \sqcup 1 \iff y^{\times} \mathbin{\square} x \sqcup 1 \sqsubseteq y^{\times} \implies x^{\times} \sqsubseteq y^{\times} \ .$$

Most of the time, this property will be used without explicit mention.

Remark 19. Like for the corresponding unfolding law (14) in KA, the following symmetric version of (29),

$$x^{\times} = x \mathbin{\square} x^{\times} \sqcup 1 \ , \tag{34}$$

is derivable from these axioms. Indeed,

$x^{\times} \sqsubseteq x \mathbin{\square} x^{\times} \sqcup 1$

$\Longleftarrow$ ⟨ by (31) and (26) ⟩

$x \mathbin{\square} (x \mathbin{\square} x^{\times} \sqcup 1) \sqcup 1 \sqsubseteq x \mathbin{\square} x^{\times} \sqcup 1$

$\Longleftarrow$ ⟨ monotonicity of $\square$ and $\sqcup$ ⟩

$x \mathbin{\square} x^{\times} \sqcup 1 \sqsubseteq x^{\times}$ —this is the other inequality we have to show

$\iff$ ⟨ by (29) ⟩

$x \mathbin{\square} x^{\times} \sqcup 1 \sqsubseteq x^{\times} \mathbin{\square} x \sqcup 1$

$\Longleftarrow$ ⟨ monotonicity of $\sqcup$ ⟩

$x \mathbin{\square} x^{\times} \sqsubseteq x^{\times} \mathbin{\square} x$

$\Longleftarrow$ ⟨ by (32) ⟩

$x^{\times} \mathbin{\square} x \mathbin{\square} x \sqcup x \sqsubseteq x^{\times} \mathbin{\square} x$

$\iff$ ⟨ by (29), (28), (26) and (30) ⟩

true .

One can show $x^{\times} = \mu_{\sqsubseteq}(y :: y \mathbin{\square} x \sqcup 1)$ with (26), (29) and (32) and $x^{\times} = \mu_{\sqsubseteq}(y :: x \mathbin{\square} y \sqcup 1)$ with (26), (34) and (31).

3.2 Demonic Algebra with Tests

Now comes the first extension of DA, demonic algebra with tests. This extension has a concept of tests like the one in KAT and it also adds the conditional operator $\sqcap_t$. In KAT, $+$ and $\cdot$ are respectively the join and meet operators of the Boolean lattice of tests. But in DAT, it will turn out that for any tests s and t, $s \sqcup t = s \mathbin{\square} t$, and that $\sqcup$ and $\square$ both act as the join operator on tests (this is also the case for the KAD-based definition of these operators given in Section 2, as can be checked). Introducing $\sqcap_t$ provides a way to express the meet of tests, as will be shown below. Here is how we deal with tests in a demonic world.

Definition 20 (Demonic algebra with tests). *A* demonic algebra with tests *(DAT) is a structure* $(A_\mathcal{D}, B_\mathcal{D}, \sqcup, \square, {}^\times, 0, 1, \sqcap_\bullet)$ *such that*

1. $(A_\mathcal{D}, \sqcup, \square, {}^\times, 0, 1)$ *is a DA;*
2. $\{1, 0\} \subseteq B_\mathcal{D} \subseteq A_\mathcal{D}$*;*
3. *for all* $t \in B_\mathcal{D}$, $1 \sqsubseteq t$*;*
4. $\sqcap_\bullet$ *is a ternary operator of type* $B_\mathcal{D} \times A_\mathcal{D} \times A_\mathcal{D} \to A_\mathcal{D}$ *that can be thought of as a family of binary operators. For each* $t \in B_\mathcal{D}$, $\sqcap_t$ *is an operator of type* $A_\mathcal{D} \times A_\mathcal{D} \to A_\mathcal{D}$*, and of type* $B_\mathcal{D} \times B_\mathcal{D} \to B_\mathcal{D}$ *if its two arguments belong to* $B_\mathcal{D}$*;*
5. $\sqcap_\bullet$ *satisfies the following properties for all* $s, t \in B_\mathcal{D}$ *and all* $x, y, z \in A_\mathcal{D}$*. In these axioms, we use the negation operator* $\neg$*, defined by*

$$\neg t = 0 \sqcap_t 1 \ . \tag{35}$$

$$x \sqcap_t y = y \sqcap_{\neg t} x \tag{36}$$
$$(t \square x) \sqcap_t y = x \sqcap_t y \tag{37}$$
$$x \sqcap_t x = x \tag{38}$$
$$x \sqcap_t 0 = t \square x \tag{39}$$
$$(x \sqcap_t y) \square z = x \square z \sqcap_t y \square z \tag{40}$$
$$s \square (x \sqcap_t y) = s \square x \sqcap_t s \square y \tag{41}$$
$$x \sqcap_t (y \sqcup z) = (x \sqcap_t y) \sqcup (x \sqcap_t z) \tag{42}$$
$$x \sqcup (y \sqcap_t z) = (x \sqcup y) \sqcap_t (x \sqcup z) \tag{43}$$
$$t \sqcup \neg t = 0 \tag{44}$$
$$\neg(1 \sqcap_t s) = \neg t \sqcup \neg s \tag{45}$$

The elements in $B_\mathcal{D}$ *are called (demonic) tests.*

Remark 21. By point 4 of the definition, $B_\mathcal{D}$ is closed under $\sqcap_\bullet$. By (35), $B_\mathcal{D}$ is closed under $\neg$ since only $\sqcap_\bullet$ is used for its definition. $B_\mathcal{D}$ is closed under $\sqcup$ and $\square$ too and this comes respectively from Proposition 22-2 and Proposition 22-8 below.

The axioms for $\sqcap_t$ given in the definition of DAT are all satisfied by the choice operator $\_ \lhd t \rhd \_$ of Hoare et al. [14,15]. The conditional operator satisfies a lot of additional laws, as shown by the following proposition, and more can be found in the precursor paper [23] (with a different syntax).

We list the correspondence between the axioms of DAT and properties of Hoare et al.'s conditional operator, using the same notation as the authors.

DAT	Laws of programming [14]	UTP [15]
$x \sqsubseteq y \iff x \sqcup y = y$	$P \subseteq Q \iff P \cup Q = Q$	$[P \Rightarrow Q] \iff [P \sqcap Q = Q]$
$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$	$P \cup (Q \cup R) = (P \cup Q) \cup R$	$P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R$
$x \sqcup y = y \sqcup x$	$P \cup Q = Q \cup P$	$P \sqcap Q = Q \sqcap P$
$x \sqcup x = x$	$P \cup P = P$	$P \sqcap P = P$
$0 \sqcup x = 0$	$\bot \cup P = \bot$	$\mathsf{true} \sqcap P = \mathsf{true}$
$x \square (y \square z) = (x \square y) \square z$	$P; (Q; R) = (P; Q); R$	$P; (Q; R) = (P; Q); R$
$0 \square x = x \square 0 = 0$	$\bot; P = P; \bot = \bot$	$\mathsf{true}; P = P; \mathsf{true} = \mathsf{true}$
$1 \square x = x \square 1 = x$	$I\!I; P = P; I\!I = P$	$I\!I_{\alpha P}; P = P; I\!I_{\alpha P} = P$
$x \square (y \sqcup z) = x \square y \sqcup x \square z$	$P; (Q \cup R) = (P; Q) \cup (P; R)$	$P; (Q \sqcap R) = (P; Q) \sqcap (P; R)$
$(x \sqcup y) \square z = x \square z \sqcup y \square z$	$(P \cup Q); R = (P; R) \cup (Q; R)$	$(P \sqcap Q); R = (P; R) \sqcap (Q; R)$
$x \sqcap_t y = y \sqcap_{\neg t} x$	$P \lhd b \rhd Q = Q \lhd \neg b \rhd P$	$P \lhd b \rhd Q = Q \lhd \neg b \rhd P$
$x \sqcap_t x = x$	$P \lhd b \rhd P = P$	$P \lhd b \rhd P = P$
$(x \sqcap_t y) \square z = x \square z \sqcap_t y \square z$	$(P \lhd b \rhd Q); R = (P; R) \lhd b \rhd (Q; R)$	$(P \lhd b \rhd Q); R = (P; R) \lhd b \rhd (Q; R)$
$x^{\times}$		$\nu R \bullet (P; R \sqcap I\!I_{\alpha(P;R)})$

We now prove some additional properties of $\sqcap_t$.

Proposition 22. *The following properties are true for all $s, t \in B_{\mathcal{D}}$ and all $x, x_1, x_2, y, y_1, y_2, z \in A_{\mathcal{D}}$.*

1. $\neg\neg t = t$
2. $s \sqcup t \in B_{\mathcal{D}}$
3. $x \sqsubseteq y \implies x \sqcap_t z \sqsubseteq y \sqcap_t z$
4. $x \sqsubseteq y \implies z \sqcap_t x \sqsubseteq z \sqcap_t y$
5. $0 \sqcap_t x = \neg t \square x$
6. $x \sqcap_t \neg t \square y = x \sqcap_t y$
7. $t \square t = t$
8. $s \sqcup t = s \square t$
9. $t \square \neg t = \neg t \square t = 0$
10. $s \square t = t \square s$
11. $\neg 1 = 0$
12. $\neg 0 = 1$
13. $x \sqsubseteq t \square x$ *and* $x \sqsubseteq x \square t$
14. $x \sqsubseteq t \square y \iff t \square x \sqsubseteq t \square y$
15. $t \square x \sqsubseteq x \iff 0 \sqsubseteq \neg t \square x$
16. $s \sqsubseteq t \implies \neg t \sqsubseteq \neg s$
17. $x \sqsubseteq y \iff t \square x \sqsubseteq t \square y \wedge \neg t \square x \sqsubseteq \neg t \square y$
18. $x = y \iff t \square x = t \square y \wedge \neg t \square x = \neg t \square y$
19. $t \square (x \sqcap_t y) = t \square x$
20. $\neg t \square (x \sqcap_t y) = \neg t \square y$
21. $x \sqsubseteq y \sqcap_t z \iff x \sqsubseteq t \square y \wedge x \sqsubseteq \neg t \square z$
22. $x \sqcap_t y \sqsubseteq z \iff x \sqsubseteq t \square z \wedge y \sqsubseteq \neg t \square z$
23. $(x_1 \sqcap_s y_1) \sqcap_t (x_2 \sqcap_s y_2) = (x_1 \sqcap_t x_2) \sqcap_s (y_1 \sqcap_t y_2)$

Proof.

1. $\neg(\neg t)$
   $=$ $\langle$ by (35) $\rangle$
   $0 \sqcap_{\neg t} 1$
   $=$ $\langle$ by (36) $\rangle$
   $1 \sqcap_t 0$
   $=$ $\langle$ by (39) $\rangle$
   t
2. $s \sqcup t$
   $=$ $\langle$ by Proposition 22-1 $\rangle$
   $\neg(\neg s) \sqcup \neg(\neg t)$
   $=$ $\langle$ by (45) $\rangle$
   $\neg(1 \sqcap_{\neg s} \neg t)$
   $\in$ $\langle$ since $\neg s \in B_{\mathcal{D}}$ and $\neg t \in B_{\mathcal{D}}$ for all $s, t \in B_{\mathcal{D}}$, and by the typing of $\sqcap_\bullet$ $\rangle$
   $B_{\mathcal{D}}$
3. $x \sqsubseteq y$
   $\Longleftrightarrow$ $\langle$ by (30) $\rangle$
   $x \sqcup y = y$
   $\Longrightarrow$ $\langle$ Leibniz $\rangle$
   $(x \sqcup y) \sqcap_t z = y \sqcap_t z$
   $\Longleftrightarrow$ $\langle$ by (36) $\rangle$
   $z \sqcap_{\neg t} (x \sqcup y) = y \sqcap_t z$
   $\Longleftrightarrow$ $\langle$ by 42 $\rangle$
   $(z \sqcap_{\neg t} x) \sqcup (z \sqcap_{\neg t} y) = y \sqcap_t z$
   $\Longleftrightarrow$ $\langle$ by (36) $\rangle$
   $(x \sqcap_t z) \sqcup (y \sqcap_t z) = y \sqcap_t z$
   $\Longleftrightarrow$ $\langle$ by (30) $\rangle$
   $x \sqcap_t z \sqsubseteq y \sqcap_t z$
4. $x \sqsubseteq y$
   $\Longrightarrow$ $\langle$ by Proposition 22-3 $\rangle$
   $x \sqcap_{\neg t} z \sqsubseteq y \sqcap_{\neg t} z$
   $\Longleftrightarrow$ $\langle$ by (36) $\rangle$
   $z \sqcap_t x \sqsubseteq z \sqcap_t y$
5. $0 \sqcap_t x$
   $=$ $\langle$ by (36) $\rangle$

$$
\begin{array}{ll}
& x \sqcap_{\neg t} 0 \\
= & \quad \langle \text{ by (39) } \rangle \\
& \neg t \square x
\end{array}
$$

6.

$$
\begin{array}{ll}
& x \sqcap_t \neg t \square y \\
= & \quad \langle \text{ by (36) } \rangle \\
& \neg t \square y \sqcap_{\neg t} x \\
= & \quad \langle \text{ by (37) } \rangle \\
& y \sqcap_{\neg t} x \\
= & \quad \langle \text{ by (36) } \rangle \\
& x \sqcap_t y
\end{array}
$$

7.

$$
\begin{array}{ll}
& t \square t \\
= & \quad \langle \text{ by (39) } \rangle \\
& t \sqcap_t 0 \\
= & \quad \langle \text{ by (44) } \rangle \\
& t \sqcap_t (t \sqcup \neg t) \\
= & \quad \langle \text{ by (42) } \rangle \\
& (t \sqcap_t t) \sqcup (t \sqcap_t \neg t) \\
= & \quad \langle \text{ by (38) } \rangle \\
& t \sqcup (t \sqcap_t \neg t) \\
= & \quad \langle \text{ by (37) } \rangle \\
& t \sqcup (1 \sqcap_t \neg t) \\
= & \quad \langle \text{ by Proposition 22-6 } \rangle \\
& t \sqcup (1 \sqcap_t 1) \\
= & \quad \langle \text{ by (38) } \rangle \\
& t \sqcup 1 \\
= & \quad \langle \text{ by Definition 20-3 } \rangle \\
& t
\end{array}
$$

8. Definition 20 gives $1 \sqsubseteq s$ from which $t \sqsubseteq t \square s$. We have $s \sqsubseteq s \square t$ and $t \sqsubseteq s \square t$ the same way. We then deduce $s \sqcup t \sqsubseteq s \square t$. We now look for $s \square t \sqsubseteq s \sqcup t$.

$$
\begin{array}{ll}
& s \square t \\
\sqsubseteq & \quad \langle \text{ because } s \sqsubseteq s \sqcup t \text{ and } t \sqsubseteq s \sqcup t \rangle \\
& (s \sqcup t) \square (s \sqcup t) \\
= & \quad \langle \text{ by Propositions 22-2 and 22-7 } \rangle \\
& s \sqcup t
\end{array}
$$

9. This follows from Proposition 22-8 and (44).

10. $s \Box t$

$=$ $\langle$ by Proposition 22-8 $\rangle$

$s \sqcup t$

$=$ $\langle$ by (21) $\rangle$

$t \sqcup s$

$=$ $\langle$ by Proposition 22-8 $\rangle$

$t \Box s$

11. $\neg 1$

$=$ $\langle$ by (35) $\rangle$

$0 \sqcap_1 1$

$=$ $\langle$ by Proposition 22-6 $\rangle$

$0 \sqcap_1 \neg 1 \Box 1$

$=$ $\langle$ by Proposition 22-9 $\rangle$

$0 \sqcap_1 0$

$=$ $\langle$ by (38) $\rangle$

0

12. This is direct from Propositions 22-1 and 22-11.

13. This follows from (26), Definition 20-3 and monotonicity of $\Box$.

14. $x \sqsubseteq t \Box y$

$\Longrightarrow$ $\langle$ left composition with t and monotonicity of $\Box$ $\rangle$

$t \Box x \sqsubseteq t \Box t \Box y$

$\Longleftrightarrow$ $\langle$ by Proposition 22-7 $\rangle$

$t \Box x \sqsubseteq t \Box y$

$\Longrightarrow$ $\langle$ by Proposition 22-13 and transitivity of $\sqsubseteq$ $\rangle$

$x \sqsubseteq t \Box y$

15. $t \Box x \sqsubseteq x$

$\Longrightarrow$ $\langle$ left composition by $\neg t$ and monotonicity of $\Box$ $\rangle$

$\neg t \Box t \Box x \sqsubseteq \neg t \Box x$

$\Longleftrightarrow$ $\langle$ by Proposition 22-9 and by (25) $\rangle$

$0 \sqsubseteq \neg t \Box x$

$\Longrightarrow$ $\langle$ by Proposition 22-4 $\rangle$

$x \sqcap_t 0 \sqsubseteq x \sqcap_t \neg t \Box x$

$\Longleftrightarrow$ $\langle$ by (39) and Proposition 22-6 $\rangle$

$t \Box x \sqsubseteq x \sqcap_t x$

$\Longleftrightarrow$ $\langle$ by (38) $\rangle$

$t \Box x \sqsubseteq x$

16. $s \sqsubseteq t$

$\Longleftrightarrow$ $\langle$ by (30) $\rangle$

$s \sqcup t = t$

$\Longleftrightarrow$ $\langle$ by Proposition 22-8 $\rangle$

$s \square t = t$

$\Longleftrightarrow$ $\langle$ by Propositions 22-15 and 22-13 $\rangle$

$0 \sqsubseteq \neg s \square t$

$\Longleftrightarrow$ $\langle$ by Proposition 22-10 $\rangle$

$0 \sqsubseteq t \square \neg s$

$\Longleftrightarrow$ $\langle$ by Proposition 22-1 $\rangle$

$0 \sqsubseteq \neg\neg t \square \neg s$

$\Longleftrightarrow$ $\langle$ by Propositions 22-15 and 22-13 $\rangle$

$\neg t \square \neg s = \neg s$

$\Longleftrightarrow$ $\langle$ by Proposition 22-8 $\rangle$

$\neg t \sqcup \neg s = \neg s$

$\Longleftrightarrow$ $\langle$ by (30) $\rangle$

$\neg t \sqsubseteq \neg s$

17. $x \sqsubseteq y$

$\Longrightarrow$ $\langle$ left composition with t and $\neg t$, and monotonicity of $\square$ $\rangle$

$t \square x \sqsubseteq t \square y \;\wedge\; \neg t \square x \sqsubseteq \neg t \square y$

$\Longrightarrow$ $\langle$ by Proposition 22-3 $\rangle$

$t \square x \sqcap_t \neg t \square x \sqsubseteq t \square y \sqcap_t \neg t \square x \;\wedge\; \neg t \square x \sqcap_{\neg t} t \square y \sqsubseteq \neg t \square y \sqcap_{\neg t} t \square y$

$\Longleftrightarrow$ $\langle$ by (36) $\rangle$

$t \square x \sqcap_t \neg t \square x \sqsubseteq t \square y \sqcap_t \neg t \square x \;\wedge\; t \square y \sqcap_t \neg t \square x \sqsubseteq t \square y \sqcap_t \neg t \square y$

$\Longrightarrow$ $\langle$ transitivity of $\sqsubseteq$ $\rangle$

$t \square x \sqcap_t \neg t \square x \sqsubseteq t \square y \sqcap_t \neg t \square y$

$\Longrightarrow$ $\langle$ by (37) and Proposition 22-6 $\rangle$

$x \sqcap_t x \sqsubseteq y \sqcap_t y$

$\Longrightarrow$ $\langle$ by (38) $\rangle$

$x \sqsubseteq y$

18. $x = y$

$\Longleftrightarrow$ $\langle$ because $\sqsubseteq$ is a partial ordering $\rangle$

$x \sqsubseteq y \;\wedge\; y \sqsubseteq x$

$\Longleftrightarrow$ $\langle$ by Proposition 22-17 $\rangle$

$t \square x \sqsubseteq t \square y \;\wedge\; \neg t \square x \sqsubseteq \neg t \square y \;\wedge\; t \square y \sqsubseteq t \square x \;\wedge\; \neg t \square y \sqsubseteq \neg t \square x$

$\Longleftrightarrow$ ⟨ because $\sqsubseteq$ is a partial ordering ⟩

$t\Box x = t\Box y \;\wedge\; \neg t\Box x = \neg t\Box y$

19. $t\Box(x \sqcap_t y)$

$=$ ⟨ by (41) ⟩

$t\Box x \sqcap_t t\Box y$

$=$ ⟨ by (37) and Proposition 22-6 ⟩

$x \sqcap_t \neg t\Box t\Box y$

$=$ ⟨ by Proposition 22-9 ⟩

$x \sqcap_t 0$

$=$ ⟨ by (39) ⟩

$t\Box x$

20. $\neg t\Box(x \sqcap_t y)$

$=$ ⟨ by (36) ⟩

$\neg t\Box(y \sqcap_{\neg t} x)$

$=$ ⟨ by Proposition 22-19 ⟩

$\neg t\Box y$

21. $x \sqsubseteq y \sqcap_t z$

$\Longleftrightarrow$ ⟨ by Proposition 22-17 ⟩

$t\Box x \sqsubseteq t\Box(y \sqcap_t z) \;\wedge\; \neg t\Box x \sqsubseteq \neg t\Box(y \sqcap_t z)$

$\Longleftrightarrow$ ⟨ by Propositions 22-19 and 22-20 ⟩

$t\Box x \sqsubseteq t\Box y \;\wedge\; \neg t\Box x \sqsubseteq \neg t\Box z$

$\Longleftrightarrow$ ⟨ by Proposition 22-14 ⟩

$x \sqsubseteq t\Box y \;\wedge\; x \sqsubseteq \neg t\Box z$

22. $x \sqcap_t y \sqsubseteq z$

$\Longleftrightarrow$ ⟨ by Proposition 22-17 ⟩

$t\Box(x \sqcap_t y) \sqsubseteq t\Box z \;\wedge\; \neg t\Box(x \sqcap_t y) \sqsubseteq \neg t\Box z$

$\Longleftrightarrow$ ⟨ by (36) ⟩

$t\Box(x \sqcap_t y) \sqsubseteq t\Box z \;\wedge\; \neg t\Box(y \sqcap_{\neg t} x) \sqsubseteq \neg t\Box z$

$\Longleftrightarrow$ ⟨ by Proposition 22-19 ⟩

$t\Box x \sqsubseteq t\Box z \;\wedge\; \neg t\Box y \sqsubseteq \neg t\Box z$

$\Longleftrightarrow$ ⟨ by Proposition 22-14 ⟩

$x \sqsubseteq t\Box z \;\wedge\; y \sqsubseteq \neg t\Box z$

23. $(x_1 \sqcap_s y_1) \sqcap_t (x_2 \sqcap_s y_2) \sqsubseteq z$

$\Longleftrightarrow$ ⟨ by Proposition 22-22 ⟩

$x_1 \sqcap_s y_1 \sqsubseteq t\Box z \;\wedge\; x_2 \sqcap_s y_2 \sqsubseteq \neg t\Box z$

$\Longleftrightarrow$ ⟨ by Proposition 22-22 ⟩

$$
\begin{aligned}
& x_1 \sqsubseteq s \Box t \Box z \ \wedge \ x_2 \sqsubseteq s \Box \neg t \Box z \ \wedge \ y_1 \sqsubseteq \neg s \Box t \Box z \ \wedge \ y_2 \sqsubseteq \neg s \Box \neg t \Box z \\
\Longleftrightarrow \quad & \qquad \langle \text{ by Proposition 22-10 } \rangle \\
& x_1 \sqsubseteq t \Box s \Box z \ \wedge \ x_2 \sqsubseteq \neg t \Box s \Box z \ \wedge \ y_1 \sqsubseteq t \Box \neg s \Box z \ \wedge \ y_2 \sqsubseteq \neg t \Box \neg s \Box z \\
\Longleftrightarrow \quad & \qquad \langle \text{ by Proposition 22-22 } \rangle \\
& x_1 \sqcap_t x_2 \sqsubseteq s \Box z \ \wedge \ y_1 \sqcap_t y_2 \sqsubseteq \neg s \Box z \\
\Longleftrightarrow \quad & \qquad \langle \text{ by Proposition 22-22 } \rangle \\
& (x_1 \sqcap_t x_2) \sqcap_s (y_1 \sqcap_t y_2) \sqsubseteq z
\end{aligned}
$$

□

Note that Propositions 22-3 and 22-4 simply express the monotonicity of $\sqcap_t$ in its two arguments. On the other hand, $\sqcap_\bullet$ is not monotonic with respect to its test argument.

As a direct consequence of Proposition 22, one can deduce the next corollary.

Corollary 23. *The set $B_\mathcal{D}$ of demonic tests forms a Boolean algebra with bottom* 1 *and top* 0*. The supremum of s and t is $s \sqcup t$ (or $s \Box t$), their infimum is $1 \sqcap_s t$ —in particular, $1 \sqcap_s \neg s = 1$—, and the negation of t is $\neg t = 0 \sqcap_t 1$ (see (35)).*

Thus, tests have quite similar properties in KAT and DAT. But there are important differences. The first one is that $\sqcup$ and $\Box$ behave the same way on tests (Proposition 22-8). The second one concerns Laws 17 and 18 of Proposition 22, which show how a proof of refinement or equality can be done by case analysis by decomposing it with cases t and $\neg t$. The same is true in KAT. However, in KAT, this decomposition can also be done on the right side, since for instance the law $x \le y \iff x \cdot t \le y \cdot t \wedge x \cdot \neg t \le y \cdot \neg t$ holds, while the corresponding law does not hold in DAT. In DAT, there is an asymmetry between left and right that can be traced back to laws (40) and (41). In (40), left distributivity holds for arbitrary elements, while right distributivity in (41) holds only for tests. Another law worth noting is Proposition 22-15. On the left of the equivalence, t acts as a *left preserver* of x and on the right, $\neg t$ acts as a *left annihilator*.

3.3 Demonic Algebra with Domain

The next extension consists in adding a domain operator to DAT. It is denoted by the symbol $\ulcorner$.

Definition 24 (Demonic algebra with domain). *A* demonic algebra with domain *(DAD) is a structure $(A_\mathcal{D}, B_\mathcal{D}, \sqcup, \Box, ^\times, 0, 1, \sqcap_\bullet, \ulcorner)$, where $(A_\mathcal{D}, B_\mathcal{D}, \sqcup, \Box, ^\times, 0, 1, \sqcap_\bullet)$ is a DAT, and the demonic domain operator $\ulcorner : A_\mathcal{D} \to B_\mathcal{D}$ satisfies the following properties for all $t \in B_\mathcal{D}$ and all $x, y \in A_\mathcal{D}$.*

$$\ulcorner(x \Box t) \Box x = x \Box t \tag{46}$$

$$\ulcorner(x \Box y) = \ulcorner(x \Box \ulcorner y) \tag{47}$$

$$\ulcorner(x \sqcup y) = \ulcorner x \sqcup \ulcorner y \tag{48}$$

Remark 25. As noted above, the axiomatisation of DA is very similar to that of KA, so one might expect the resemblance to continue between DAD and KAD. In particular, looking at the angelic version of Definition 24, namely Definition 4, one might expect to find axioms like $\ulcorner x \,\Box\, x \sqsubseteq x$ and $t \sqsubseteq \ulcorner(t \,\Box\, x)$, or equivalently, $t \sqsubseteq \ulcorner x \iff t \,\Box\, x \sqsubseteq x$. These three properties can be derived from the chosen axioms (see Propositions 29-2, 29-5 and 29-6) but (46) cannot be derived from them, even when assuming (47) and (48). But (46) holds in KAD-based demonic algebras. Since our goal is to come as close as possible to these, we include (46) as an axiom.

Example 26. For this example $A_{\mathcal{D}} = \{0, \mathsf{s}, \mathsf{t}, 1, \mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}\}$ and $B_{\mathcal{D}} = \{0, \mathsf{s}, \mathsf{t}, 1\}$. The demonic operators are defined by the following tables.

$\sqcup$	0	s	t	1	a	b	c	d
0	0	0	0	0	0	0	0	0
s	0	s	0	s	a	a	a	a
t	0	0	t	t	0	0	t	t
1	0	s	t	1	a	a	c	c
a	0	a	0	a	a	a	a	a
b	0	a	0	a	a	b	a	b
c	0	a	t	c	a	a	c	c
d	0	a	t	c	a	b	c	d

$\Box$	0	s	t	1	a	b	c	d
0	0	0	0	0	0	0	0	0
s	0	s	0	s	a	b	a	b
t	0	0	t	t	0	0	t	t
1	0	s	t	1	a	b	c	d
a	0	a	0	a	a	a	a	a
b	0	a	0	b	a	a	a	a
c	0	a	t	c	a	a	c	c
d	0	a	t	d	a	a	c	c

	$\times$
0	0
s	s
t	t
1	1
a	a
b	a
c	c
d	c

	$\neg$
0	1
s	t
t	s
1	0

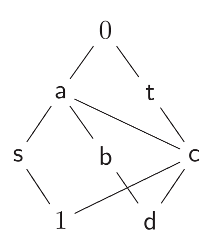
	$\ulcorner$
0	0
s	s
t	t
1	1
a	s
b	s
c	1
d	1

$x \sqcap_0 y = y$ $\qquad$ $x \sqcap_1 y = x$

$\sqcap_{\mathsf{s}}$	0	s	t	1	a	b	c	d
0	0	0	t	t	0	0	t	t
s	s	s	1	1	s	s	1	1
t	0	0	t	t	0	0	t	t
1	1	1	1	1	1	1	1	1
a	a	a	c	c	a	a	c	c
b	b	b	d	d	b	b	d	d
c	a	a	c	c	a	a	c	c
d	b	b	d	d	b	b	d	d

$\sqcap_{\mathsf{t}}$	0	s	t	1	a	b	c	d
0	0	s	0	s	a	b	a	b
s	0	s	0	s	a	b	a	b
t	t	1	t	1	c	d	c	d
1	t	1	t	1	c	d	c	d
a	0	s	0	s	a	b	a	b
b	0	s	0	s	a	b	a	b
c	t	1	t	1	c	d	c	d
d	t	1	t	1	c	d	c	d

The demonic refinement ordering corresponding to $\sqcup$ is represented in the following semilattice.



This algebra is a DAT for which $\ulcorner x \,\Box\, x \sqsubseteq x$, $t \sqsubseteq \ulcorner(t \,\Box\, x)$, (47) and (48) all hold, but (46) does not. Indeed $\ulcorner(\mathsf{b} \,\Box\, \mathsf{s}) \,\Box\, \mathsf{b} = \mathsf{a} \neq \mathsf{b} = \mathsf{b} \,\Box\, \mathsf{s}$.

Then why choose (46) rather than $\ulcorner\!\!\ulcorner x \Box x \sqsubseteq x$ and $t \sqsubseteq \ulcorner\!\!\ulcorner(t \Box x)$? The justification is twofold. Firstly, models that come from KAD satisfy property (46), that is, $\ulcorner(x \Box_A t) \Box_A x = x \Box_A t$ (see Theorem 17-1). Secondly, there are strong indications that this law will be needed to solve Conjecture 43 (see page 45).

Law (47) is locality in a demonic world.

In KAD, it is not necessary to have an axiom like (48), because additivity of $\ulcorner$ (Proposition 6-11) follows from (6) and the laws of KAT. However, it is necessary in the context of demonic algebras since the following example satisfies all prescribed laws except that one.

Example 27. For this example $A_{\mathcal{D}} = \{0, 1, \mathsf{a}\}$ and $B_{\mathcal{D}} = \{0, 1\}$. The demonic operators are defined by the following tables.

$\sqcup$	0	1	a
0	0	0	0
1	0	1	0
a	0	0	a

$\Box$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	1

	$^\times$
0	0
1	1
a	0

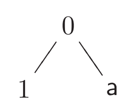
$\sqcap_0$	0	1	a
0	0	1	a
1	0	1	a
a	0	1	a

$\sqcap_1$	0	1	a
0	0	0	0
1	1	1	1
a	a	a	a

	$\neg$
0	1
1	0

	$\ulcorner\!\!\ulcorner$
0	0
1	1
a	1

The demonic refinement ordering corresponding to $\sqcup$ is represented in the following semilattice.



This algebra is a DAT and, in addition, (46) and (47) are satisfied, but (48) is not. Indeed $\ulcorner\!\!\ulcorner(1 \sqcup \mathsf{a}) \neq \ulcorner\!\!\ulcorner 1 \sqcup \ulcorner\!\!\ulcorner \mathsf{a}$.

Examples 26 and 27 show that Axioms (46) and (48) are independent from each other and also from (47). The following example completes this proof of independence. Thus, the three axioms that define demonic domain are independent.

Example 28. For this example $A_{\mathcal{D}} = \{0, 1, \mathsf{a}\}$ and $B_{\mathcal{D}} = \{0, 1\}$. The demonic operators are defined by the following tables.

$\sqcup$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

$\Box$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

	$^\times$
0	0
1	1
a	0

$\sqcap_0$	0	1	a
0	0	1	a
1	0	1	a
a	0	1	a

$\sqcap_1$	0	1	a
0	0	0	0
1	1	1	1
a	a	a	a

	$\neg$
0	1
1	0

	$\ulcorner\!\!\ulcorner$
0	0
1	1
a	1

The Hasse diagram of the demonic refinement ordering corresponding to $\sqcup$ is simply given by $1 \sqsubset \mathsf{a} \sqsubset 0$. In this DAT, (46) and (48) are satisfied, but (47) is not. Indeed $\ulcorner\!\!\ulcorner(\mathsf{a} \Box \mathsf{a}) = 0 \neq 1 = \ulcorner\!\!\ulcorner(\mathsf{a} \Box \ulcorner\!\!\ulcorner \mathsf{a})$.

By Proposition 29-2 below, $\ulcorner\!\!\ulcorner x$ is a left preserver of x. By Proposition 29-6, it is the greatest left preserver. Similarly, by Proposition 29-9, $\neg\ulcorner\!\!\ulcorner x$ is a left annihilator of x. By Proposition 29-8, it is the least left annihilator (since Proposition 29-8 can be rewritten as $\neg\ulcorner\!\!\ulcorner x \sqsubseteq t \iff 0 \sqsubseteq t \Box x$).

Proposition 29. *In a DAD, the demonic domain operator satisfies the following properties. Take $x, y \in A_{\mathcal{D}}$ and $t \in B_{\mathcal{D}}$.*

1. $x \sqsubseteq y \Longrightarrow \ulcorner\!\!\ulcorner x \sqsubseteq \ulcorner\!\!\ulcorner y$
2. $\ulcorner\!\!\ulcorner x \Box x = x$
3. $\ulcorner\!\!\ulcorner t = t$
4. $\ulcorner\!\!\ulcorner(t \Box x) = t \Box \ulcorner\!\!\ulcorner x$
5. $t \sqsubseteq \ulcorner\!\!\ulcorner(t \Box x)$
6. $t \sqsubseteq \ulcorner\!\!\ulcorner x \Longleftrightarrow t \Box x \sqsubseteq x$
7. $\ulcorner\!\!\ulcorner x = \mathsf{max}_{\sqsubseteq}\{t \mid t \in B_{\mathcal{D}} \wedge t \Box x = x\}$
8. $t \sqsubseteq \ulcorner\!\!\ulcorner x \iff 0 \sqsubseteq \neg t \Box x$
9. $\neg \ulcorner\!\!\ulcorner x \Box x = 0$
10. $\ulcorner\!\!\ulcorner x \sqsubseteq \ulcorner\!\!\ulcorner(x \Box y)$
11. $\ulcorner\!\!\ulcorner x = 0 \Longleftrightarrow x = 0$
12. $\ulcorner\!\!\ulcorner(x \sqcap_t y) = \ulcorner\!\!\ulcorner x \sqcap_t \ulcorner\!\!\ulcorner y$
13. $x \sqcup y = \ulcorner\!\!\ulcorner x \Box \ulcorner\!\!\ulcorner y \Box (x \sqcup y)$
14. $\ulcorner\!\!\ulcorner(x \Box s) \Box \ulcorner\!\!\ulcorner(x \Box t) = \ulcorner\!\!\ulcorner(x \Box s \Box t)$

All the above laws except 12 and 14 are identical to laws of $\ulcorner$, after compensating for the reverse ordering of the Boolean lattice (on tests, $\sqsubseteq$ corresponds to $\geq$).

Proof.

1. $x \sqsubseteq y$
   $\Longleftrightarrow$ ⟨ by (30) ⟩
   $x \sqcup y = y$
   $\Longrightarrow$ ⟨ evaluating demonic domain both sides and by (48) ⟩
   $\ulcorner\!\!\ulcorner x \sqcup \ulcorner\!\!\ulcorner y = \ulcorner\!\!\ulcorner y$
   $\Longleftrightarrow$ ⟨ by (30) ⟩
   $\ulcorner\!\!\ulcorner x \sqsubseteq \ulcorner\!\!\ulcorner y$
2. This is direct from (46) with $t := 1$ and (26).
3. This is direct from (46) with $x := 1$ and (26).
4. $t \Box \ulcorner\!\!\ulcorner x$
   $=$ ⟨ by Proposition 29-3 ⟩
   $\ulcorner\!\!\ulcorner(t \Box \ulcorner\!\!\ulcorner x)$
   $=$ ⟨ by (47) ⟩
   $\ulcorner\!\!\ulcorner(t \Box x)$
5. By Definition 20-3, and Proposition 29-4, $t = t \Box 1 \sqsubseteq t \Box \ulcorner\!\!\ulcorner x = \ulcorner\!\!\ulcorner(t \Box x)$.
6. [$\Longrightarrow$] By the assumption, monotonicity of $\Box$ and Proposition 29-1, $t \Box x \sqsubseteq \ulcorner\!\!\ulcorner x \Box x \sqsubseteq x$.

$[\Longleftarrow]$

$$t \Box x \sqsubseteq x$$

$\Longrightarrow$ ⟨ by Proposition 29-1 ⟩

$$\ulcorner(t \Box x) \sqsubseteq \ulcorner x$$

$\Longrightarrow$ ⟨ by Proposition 29-5 ⟩

$$t \sqsubseteq \ulcorner x$$

7. This is direct from Proposition 29-6.
8. $$t \sqsubseteq \ulcorner x$$

$\Longleftrightarrow$ ⟨ by Proposition 29-6 ⟩

$$t \Box x \sqsubseteq x$$

$\Longleftrightarrow$ ⟨ by Proposition 22-15 ⟩

$$0 \sqsubseteq \neg t \Box x$$

9. This law follows directly from Proposition 29-8 and (33).
10. Since $\ulcorner x \Box (x \Box y) = (\ulcorner x \Box x) \Box y = x \Box y$, the result follows from Proposition 29-6.
11. $$\ulcorner x = 0$$

$\Longleftrightarrow$ ⟨ by (33) ⟩

$$0 \sqsubseteq \ulcorner x$$

$\Longleftrightarrow$ ⟨ by Proposition 29-6 ⟩

$$0 \Box x \sqsubseteq x$$

$\Longleftrightarrow$ ⟨ by (25) ⟩

$$0 \sqsubseteq x$$

$\Longleftrightarrow$ ⟨ by (33) ⟩

$$x = 0$$

12. $$s \sqsubseteq \ulcorner(x \sqcap_t y)$$

$\Longleftrightarrow$ ⟨ by Proposition 29-6 ⟩

$$s \Box (x \sqcap_t y) \sqsubseteq x \sqcap_t y$$

$\Longleftrightarrow$ ⟨ by (41) ⟩

$$s \Box x \sqcap_t s \Box y \sqsubseteq x \sqcap_t y$$

$\Longleftrightarrow$ ⟨ by Proposition 22-22 ⟩

$$s \Box x \sqsubseteq t \Box (x \sqcap_t y) \ \wedge \ s \Box y \sqsubseteq \neg t \Box (x \sqcap_t y)$$

$\Longleftrightarrow$ ⟨ by Proposition 22-19 and (36) ⟩

$$s \Box x \sqsubseteq t \Box x \ \wedge \ s \Box y \sqsubseteq \neg t \Box y$$

$\Longleftrightarrow$ ⟨ by Proposition 22-14 ⟩

$$t \Box s \Box x \sqsubseteq t \Box x \ \wedge \ \neg t \Box s \Box y \sqsubseteq \neg t \Box y$$

$\Longleftrightarrow$ ⟨ by Proposition 22-10 ⟩

$s \square t \square x \sqsubseteq t \square x \ \wedge\ s \square \neg t \square y \sqsubseteq \neg t \square y$

$\iff$ $\langle$ by Proposition 29-6 $\rangle$

$s \sqsubseteq \ulcorner(t \square x) \ \wedge\ s \sqsubseteq \ulcorner(\neg t \square y)$

$\iff$ $\langle$ by Proposition 29-4 $\rangle$

$s \sqsubseteq t \square \ulcorner x \ \wedge\ s \sqsubseteq \neg t \square \ulcorner y$

$\iff$ $\langle$ by Proposition 22-21 $\rangle$

$s \sqsubseteq \ulcorner x \sqcap_t \ulcorner y$

13. $x \sqcup y$

$=$ $\langle$ by Proposition 29-2 $\rangle$

$\ulcorner(x \sqcup y) \square (x \sqcup y)$

$=$ $\langle$ by (48) $\rangle$

$(\ulcorner x \sqcup \ulcorner y) \square (x \sqcup y)$

$=$ $\langle$ by Proposition 22-8 $\rangle$

$\ulcorner x \square \ulcorner y \square (x \sqcup y)$

14. $\ulcorner(x \square s) \square \ulcorner(x \square t)$

$=$ $\langle$ by Proposition 22-8 $\rangle$

$\ulcorner(x \square s) \sqcup \ulcorner(x \square t)$

$=$ $\langle$ by (48) $\rangle$

$\ulcorner(x \square s) \sqcup (x \square t)$

$=$ $\langle$ by (27) $\rangle$

$\ulcorner(x \square (s \sqcup t))$

$=$ $\langle$ by Proposition 22-8 $\rangle$

$\ulcorner(x \square s \square t)$ $\square$

To simplify the notation when possible, we will use the abbreviation

$$x \sqcap y = x \sqcap_{\ulcorner x} y \ . \tag{49}$$

Under special conditions, $\sqcap$ has easy to use properties, as shown by the next corollary. The most useful cases are when $\sqcap$ is used on tests and when $\ulcorner x \square \ulcorner y = 0$.

Corollary 30. *Let x, y, z be arbitrary elements and s, t be tests of a DAD.*

1. *$s \sqcap t$ is the meet of s and t in the Boolean lattice of tests.*
2. $x \sqcap y = x \sqcap \neg\ulcorner x \square y$
3. $0 \sqcap x = x \sqcap 0 = x$
4. $t \square (x \sqcap y) = t \square x \sqcap t \square y$
5. $x = t \square x \sqcap \neg t \square x$
6. $\ulcorner x \sqsubseteq t \implies t \square (x \sqcap y) = t \square x$
7. $\neg\ulcorner x \sqsubseteq t \implies t \square (x \sqcap y) = t \square y$
8. $\ulcorner x \square y = \ulcorner y \square x \implies x \sqcap y = y \sqcap x$

9. $\ulcorner x \square \ulcorner y = 0 \implies \ulcorner x \square y = \ulcorner y \square x$
10. $x \sqcap x = x$
11. $x \sqcap y \sqsubseteq x$
12. $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
13. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
14. $\ulcorner (x \sqcap y) = \ulcorner x \sqcap \ulcorner y$
15. $\ulcorner x \square \ulcorner y = 0 \Longrightarrow (x \sqcap y) \square z = x \square z \sqcap y \square z$

Proof. 1. This follows from Corollary 23, since $1 \sqcap_s t = s \sqcap_s t = s \sqcap t$ by (37) and (49).

2.
$x \sqcap y$
$=$ ⟨ by (49) ⟩
$x \sqcap_{\ulcorner x} y$
$=$ ⟨ by Proposition 22-6 ⟩
$x \sqcap_{\ulcorner x} \neg \ulcorner x \square y$
$=$ ⟨ by (49) ⟩
$x \sqcap \neg \ulcorner x \square y$

3.
$0 \sqcap x$
$=$ ⟨ by (49) and Proposition 29-3 ⟩
$0 \sqcap_0 x$
$=$ ⟨ by Proposition 22-5 ⟩
$\neg 0 \square x$
$=$ ⟨ by Proposition 22-12 and (26) ⟩
x
$=$ ⟨ by (39) and Proposition 29-2 ⟩
$x \sqcap_{\ulcorner x} 0$
$=$ ⟨ by (49) ⟩
$x \sqcap 0$

4.
$z \sqsubseteq t \square (x \sqcap y)$
$\iff$ ⟨ by (49) ⟩
$z \sqsubseteq t \square (x \sqcap_{\ulcorner x} y)$
$\iff$ ⟨ by (41) and Proposition 22-7 ⟩
$z \sqsubseteq t \square t \square x \sqcap_{\ulcorner x} t \square y$
$\iff$ ⟨ by Propositions 22-21 and 22-10 ⟩
$z \sqsubseteq t \square \ulcorner x \square t \square x \;\wedge\; z \sqsubseteq \neg \ulcorner x \square t \square y$
$\iff$ ⟨ by (39) and Propositions 22-10, 22-7 and 22-9 ⟩
$z \sqsubseteq t \square \ulcorner x \square t \square x \;\wedge\; z \sqsubseteq (\neg \ulcorner x \square t \sqcap_t \neg t \square t) \square y$
$\iff$ ⟨ by Proposition 22-6 ⟩

$z \sqsubseteq t \square \ulcorner x \square t \square x \;\wedge\; z \sqsubseteq (\neg \ulcorner x \square t \,\overline{\sqcap}_t\, t) \square y$

$\iff$ $\langle$ by (40) and (26) $\rangle$

$z \sqsubseteq t \square \ulcorner x \square t \square x \;\wedge\; z \sqsubseteq (\neg \ulcorner x \,\overline{\sqcap}_t\, 1) \square t \square y$

$\iff$ $\langle$ by (36) $\rangle$

$z \sqsubseteq t \square \ulcorner x \square t \square x \;\wedge\; z \sqsubseteq (1 \,\overline{\sqcap}_{\neg t}\, \neg \ulcorner x) \square t \square y$

$\iff$ $\langle$ by (45) and Proposition 22-8 $\rangle$

$z \sqsubseteq t \square \ulcorner x \square t \square x \;\wedge\; z \sqsubseteq \neg (t \square \ulcorner x) \square t \square y$

$\iff$ $\langle$ by Proposition 22-21 $\rangle$

$z \sqsubseteq t \square x \,\overline{\sqcap}_{t \square \ulcorner x}\, t \square y$

$\iff$ $\langle$ by (49) and Proposition 29-9 $\rangle$

$z \sqsubseteq t \square x \,\overline{\sqcap}\, t \square y$

5. $x = t \square x \,\overline{\sqcap}\, \neg t \square x$

$\iff$ $\langle$ by Proposition 22-18 $\rangle$

$t \square x = t \square (t \square x \,\overline{\sqcap}\, \neg t \square x) \;\wedge\; \neg t \square x = \neg t \square (t \square x \,\overline{\sqcap}\, \neg t \square x)$

$\iff$ $\langle$ by Corollary 30-4, and Propositions 22-7 and 22-9 $\rangle$

$t \square x = t \square x \,\overline{\sqcap}\, 0 \;\wedge\; \neg t \square x = 0 \,\overline{\sqcap}\, \neg t \square x$

$\iff$ $\langle$ by Corollary 30-3 $\rangle$

true

6. Suppose $\ulcorner x \sqsubseteq t$.

$t \square (x \,\overline{\sqcap}\, y)$

$=$ $\langle$ by Corollaries 30-2 and 30-4 $\rangle$

$t \square x \,\overline{\sqcap}\, t \square \neg \ulcorner x \square y$

$=$ $\langle$ by hypothesis $\ulcorner x \sqsubseteq t$ so $0 \sqsubseteq t \square \neg \ulcorner x$ by Proposition 22-9 $\rangle$

$t \square x \,\overline{\sqcap}\, 0$

$=$ $\langle$ by Corollary 30-3 $\rangle$

$t \square x$

7. Suppose $\neg \ulcorner x \sqsubseteq t$.

$t \square (x \,\overline{\sqcap}\, y)$

$=$ $\langle$ by Corollary 30-4 and Proposition 29-2 $\rangle$

$t \square \ulcorner x \square x \,\overline{\sqcap}\, t \square y$

$=$ $\langle$ by hypothesis $\neg \ulcorner x \sqsubseteq t$ so $0 \sqsubseteq t \square \ulcorner x$ by Proposition 22-9 $\rangle$

$0 \,\overline{\sqcap}\, t \square y$

$=$ $\langle$ by Corollay 30-3 $\rangle$

$t \square y$

8. Suppose $\ulcorner x \,\square\, y = \ulcorner y \,\square\, x$.

$x \sqcap y = y \sqcap x$

$\iff$ $\langle$ by Proposition 22-18 and (49) $\rangle$

$\ulcorner x \,\square\, (x \sqcap y) = \ulcorner x \,\square\, (y \sqcap_{\ulcorner y} x) \;\wedge\; \neg \ulcorner x \,\square\, (x \sqcap y) = \neg \ulcorner x \,\square\, (y \sqcap x)$

$\iff$ $\langle$ by (41) and Corollaries 30-6, 30-7 and 30-4 $\rangle$

$\ulcorner x \,\square\, x = \ulcorner x \,\square\, y \sqcap_{\ulcorner y} \ulcorner x \,\square\, x \;\wedge\; \neg \ulcorner x \,\square\, y = \neg \ulcorner x \,\square\, y \sqcap \neg \ulcorner x \,\square\, x$

$\iff$ $\langle$ by Propositions 29-2 and 29-9 $\rangle$

$x = \ulcorner x \,\square\, y \sqcap_{\ulcorner y} x \;\wedge\; \neg \ulcorner x \,\square\, y = \neg \ulcorner x \,\square\, y \sqcap 0$

$\iff$ $\langle$ by Corollary 30-3 $\rangle$

$x = \ulcorner x \,\square\, y \sqcap_{\ulcorner y} x \;\wedge\; \mathsf{true}$

$\iff$ $\langle$ by hypothesis $\rangle$

$x = \ulcorner y \,\square\, x \sqcap_{\ulcorner y} x$

$\iff$ $\langle$ by (37) and (38) $\rangle$

true

9. $\ulcorner x \,\square\, \ulcorner y = 0$

$\iff$ $\langle$ by Propositions 29-11, 29-4 and 22-10 $\rangle$

$\ulcorner x \,\square\, y = 0 \wedge \ulcorner y \,\square\, x = 0$

$\implies$ $\langle$ logic $\rangle$

$\ulcorner x \,\square\, y = \ulcorner y \,\square\, x$

10. $x \sqcap x$

$=$ $\langle$ by (49) $\rangle$

$x \sqcap_{\ulcorner x} x$

$=$ $\langle$ by (38) $\rangle$

x

11. $x \sqcap y \sqsubseteq x$

$\iff$ $\langle$ by (49) $\rangle$

$x \sqcap_{\ulcorner x} y \sqsubseteq x$

$\iff$ $\langle$ by Proposition 22-22 $\rangle$

$x \sqsubseteq \ulcorner x \,\square\, x \wedge y \sqsubseteq \neg \ulcorner x \,\square\, x$

$\iff$ $\langle$ by Propositions 29-2 and 29-9 $\rangle$

$x \sqsubseteq x \wedge y \sqsubseteq 0$

$\iff$ $\langle$ by (33) $\rangle$

true

12. $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$

$\iff$ $\langle$ by Proposition 22-18 $\rangle$

$$\ulcorner x \square ((x \sqcap y) \sqcap z) = \ulcorner x \square (x \sqcap (y \sqcap z)) \ \wedge$$
$$\neg \ulcorner x \square ((x \sqcap y) \sqcap z) = \neg \ulcorner x \square (x \sqcap (y \sqcap z))$$

$\Longleftrightarrow$ ⟨ by Corollary 30-11 $x \sqcap y \sqsubseteq x$ and thus $\ulcorner(x \sqcap y) \sqsubseteq \ulcorner x$ by Proposition 29-1; then apply Corollary 30-6 twice and Corollary 30-7 once ⟩

$$\ulcorner x \square (x \sqcap y) = \ulcorner x \square x \ \wedge \ \neg \ulcorner x \square ((x \sqcap y) \sqcap z) = \neg \ulcorner x \square (y \sqcap z)$$

$\Longleftrightarrow$ ⟨ by Corollary 30-6 ⟩

$$\mathsf{true} \ \wedge \ \neg \ulcorner x \square ((x \sqcap y) \sqcap z) = \neg \ulcorner x \square (y \sqcap z)$$

$\Longleftrightarrow$ ⟨ by Corollary 30-4 ⟩

$$\neg \ulcorner x \square (x \sqcap y) \sqcap \neg \ulcorner x \square z = \neg \ulcorner x \square y \sqcap \neg \ulcorner x \square z$$

$\Longleftrightarrow$ ⟨ by Corollary 30-7 ⟩

$$\mathsf{true}$$

13. $x \sqcup (y \sqcap z)$

$=$ ⟨ by (49) ⟩

$$x \sqcup (y \sqcap_{\ulcorner y} z)$$

$=$ ⟨ by (43) ⟩

$$(x \sqcup y) \sqcap_{\ulcorner y} (x \sqcup z)$$

$=$ ⟨ by (37) and Proposition 22-6 ⟩

$$\ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner y} \neg \ulcorner y \square (x \sqcup z)$$

$=$ ⟨ by Corollary 30-5 ⟩

$$\ulcorner y \square \big(\ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner y} \neg \ulcorner y \square (x \sqcup z)\big) \ \sqcap \ \neg \ulcorner y \square \big(\ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner y} \neg \ulcorner y \square (x \sqcup z)\big)$$

$=$ ⟨ by (41) and Propositions 22-7 and 22-9 ⟩

$$\big(\ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner y} 0\big) \ \sqcap \ \big(0 \sqcap_{\ulcorner y} \neg \ulcorner y \square (x \sqcup z)\big)$$

$=$ ⟨ by (39) and Propositions 22-5 and 22-7 ⟩

$$\ulcorner y \square (x \sqcup y) \ \sqcap \ \neg \ulcorner y \square (x \sqcup z)$$

$=$ ⟨ by Corollaries 23 and 30-1, and Boolean algebra ⟩

$$\ulcorner y \square (x \sqcup y) \ \sqcap \ \neg(\ulcorner x \sqcup \ulcorner y) \square \neg \ulcorner y \square (x \sqcup z)$$

$=$ ⟨ by (39), (48) and Propositions 29-2, 22-10 22-5 ⟩

$$\big(\ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} 0\big) \ \sqcap \ \big(0 \sqcap_{\ulcorner x \sqcup \ulcorner y} \neg \ulcorner y \square (x \sqcup z)\big)$$

$=$ ⟨ by (41) and Propositions 29-13, 22-7, 22-9, 22-10 ⟩

$$\ulcorner y \square \big(\ulcorner x \square \ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} \neg \ulcorner y \square (x \sqcup z)\big) \sqcap$$
$$\neg \ulcorner y \square \big(\ulcorner x \square \ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} \neg \ulcorner y \square (x \sqcup z)\big)$$

$=$ ⟨ by Proposition 29-13 and Corollary 30-5 ⟩

$$\ulcorner x \square \ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} \neg \ulcorner y \square \ulcorner x \square \ulcorner z \square (x \sqcup z)$$

$=$ ⟨ by Corollary 30-1 and Boolean algebra ⟩

$$\ulcorner x \square \ulcorner y \square (x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} (\neg \ulcorner x \sqcap \neg \ulcorner y) \square \ulcorner x \square \ulcorner z \square (x \sqcup z)$$

$=$ ⟨ by Proposition 29-13 and Corollary 30-1 ⟩

$(x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} \neg(\ulcorner x \sqcup \ulcorner y) \square (x \sqcup z)$

$=$ ⟨ by Proposition 22-6 ⟩

$(x \sqcup y) \sqcap_{\ulcorner x \sqcup \ulcorner y} (x \sqcup z)$

$=$ ⟨ by (49) and (48) ⟩

$(x \sqcup y) \sqcap (x \sqcap z)$

14. $\ulcorner(x \sqcap y)$

$=$ ⟨ by (49) ⟩

$\ulcorner(x \sqcap_{\ulcorner x} y)$

$=$ ⟨ by Proposition 29-12 ⟩

$\ulcorner x \sqcap_{\ulcorner x} \ulcorner y$

$=$ ⟨ by (49) and Proposition 29-3 ⟩

$\ulcorner x \sqcap \ulcorner y$

15. Suppose $\ulcorner x \square \ulcorner y = 0$.

$(x \sqcap y) \square z$

$=$ ⟨ by (49) ⟩

$(x \sqcap_{\ulcorner x} y) \square z$

$=$ ⟨ by (40) ⟩

$x \square z \sqcap_{\ulcorner x} y \square z$

$=$ ⟨ by Corollary 30-5 ⟩

$\ulcorner x \square (x \square z \sqcap_{\ulcorner x} y \square z) \ \sqcap \ \neg \ulcorner x \square (x \square z \sqcap_{\ulcorner x} y \square z)$

$=$ ⟨ by (41), the assumption and Propositions 29-2, 29-9 ⟩

$(x \square z \sqcap_{\ulcorner x} 0) \ \sqcap \ (0 \sqcap_{\ulcorner x} \neg \ulcorner x \square y \square z)$

$=$ ⟨ by (39) and Propositions 22-5, 29-2, 22-7 ⟩

$x \square z \ \sqcap \ \neg \ulcorner x \square \ulcorner y \square y \square z$

$=$ ⟨ by the assumption and Boolean algebra, $\neg \ulcorner x \square \ulcorner y = \ulcorner y$, and by Proposition 29-2 ⟩

$x \square z \ \sqcap \ y \square z$ □

Remark 31. By Corollary 30-11, $x \sqcap y \sqsubseteq x$. In general, $x \sqcap y \sqsubseteq y$ does not hold. Take the relations $x = \{(0,0)\}$ and $y = \{(0,1)\}$ as a counter-example.

By Corollary 30-13 and Definition (21), $(x \sqcap y) \sqcup z = (x \sqcup z) \sqcap (y \sqcup z)$. However, $(x \sqcup y) \sqcap z = x \sqcap z \sqcup y \sqcap z$ is false in general. Take the relations $x = \{(0,0)\}$, $y = \{\}$ and $z = \{(0,1)\}$ as a counter-example. Furthermore, the equality $(x \sqcap y) \square z = x \square z \sqcap y \square z$ is also false in general (compare with (40)). Take the relations $x = \{(0,0),(0,1),(1,0),(1,1)\}$, $y = \{(0,1),(1,1)\}$ and $z = \{(1,1)\}$ as a counter-example.

Remark 32. In the sequel, some transformations based on Corollaries 23 and 30-1 are simply justified by invoking "Boolean algebra".

In KAD, it can be shown that the set of tests is maximal in the sense that, if an element s has a complement relative to 1, then it is a test [9,11]. In KAD, we say that an element y is the *complement of x relative to 1* iff $x + y = 1$ and $x \cdot y = 0$. In DAD, there are two possible definitions for the notion of complement relative to 1.

Definition 33. *We say that*

1. *x is the* $\sqcup$-complement of y relative to 1 *iff* $x \sqcap y = 1$ *and* $x \sqcup y = 0$*;*
2. *x is the* $\Box$-complement of y relative to 1 *iff* $x \sqcap y = 1$ *and* $x \Box y = 0$.

These definitions are asymmetric, because $x \sqcap y$ and $x \Box y$ need not be equal to $y \sqcap x$ and $y \Box x$, respectively, but, as simply follows from the following theorem, it nevertheless turns out that the two definitions are both equivalent to $x \sqcap y = y \sqcap x = 1 \;\wedge\; x \Box y = y \Box x = 0$. The theorem also shows that the maximality result of KAD also holds in DAD.

Theorem 34. *Let $\mathcal{D} = (A_{\mathcal{D}}, B_{\mathcal{D}}, \sqcup, \Box, {}^{\times}, 0, 1, \sqcap_{\bullet}, \ulcorner\,)$ be a DAD and let $x, y \in A_{\mathcal{D}}$.*

1. *$x \sqcap y = 1 \implies \ulcorner x = x$.*
2. *$x \sqcap y = 1 \wedge x \sqcup y = 0 \implies \ulcorner y = y$.*
3. *x is the $\sqcup$-complement of y iff y is the $\sqcup$-complement of x.*
4. *x is the $\Box$-complement of y iff x is the $\sqcup$-complement of y.*
5. *x is the $\Box$-complement of y iff y is the $\Box$-complement of x.*
6. *$\ulcorner x = x \iff x \in B_{\mathcal{D}}$.*
7. *The set $B_{\mathcal{D}}$ consists of all the elements that have a ($\sqcup$ or $\Box$)-complement relative to 1.*

Proof.

1. Using the hypothesis, (26), Corollary 30-6 and Proposition 29-2, we get

$$\ulcorner x = \ulcorner x \Box (x \sqcap y) = \ulcorner x \Box x = x \ .$$

2. Assume $x \sqcap y = 1$ and $x \sqcup y = 0$. We show $\ulcorner y = y$.

$$
\begin{array}{ll}
 & \ulcorner y \\
= & \quad\langle \text{ by hypothesis and (26) } \rangle \\
 & \ulcorner y \Box (x \sqcap y) \\
= & \quad\langle \text{ by Corollary 30-4 and Proposition 29-2 } \rangle \\
 & \ulcorner y \Box \ulcorner x \Box x \,\sqcap\, \ulcorner y \Box y \\
= & \quad\langle \text{ by Propositions 29-2 and 22-8 } \rangle \\
 & (\ulcorner y \sqcup \ulcorner x) \Box x \sqcap y
\end{array}
$$

$=$ $\quad\langle$ by (48) $\rangle$

$\ulcorner(x \sqcup y) \square x \sqcap y$

$=$ $\quad\langle$ by hypothesis and Proposition 29-3 $\rangle$

$0 \square x \sqcap y$

$=$ $\quad\langle$ by (25) and Corollary 30-3 $\rangle$

y

3. Assume that x is the $\sqcup$-complement of y. Then $\ulcorner x = x$ and $\ulcorner y = y$ by Definition 33-1 and Theorems 34-1 and 34-2. Thus, by Corollary 30-1 and the hypothesis,

$$y \sqcap x = \ulcorner y \sqcap \ulcorner x = \ulcorner x \sqcap \ulcorner y = x \sqcap y = 1 \ .$$

By the assumption $x \sqcup y = 0$ and (21), $y \sqcup x = 0$ and hence y is a $\sqcup$-complement of x by Definition 33-1.
The reverse implication holds by symmetry.
4. We have to show $x \sqcap y = 1 \wedge x \square y = 0 \iff x \sqcap y = 1 \wedge x \sqcup y = 0$. Assuming $x \sqcap y = 1$, we show $x \square y = 0 \iff x \sqcup y = 0$.

$x \square y = 0$

$\iff$ $\quad\langle$ by Proposition 29-11 $\rangle$

$\ulcorner(x \square y) = 0$

$\iff$ $\quad\langle$ by (47) $\rangle$

$\ulcorner(x \square \ulcorner y) = 0$

$\iff$ $\quad\langle$ by Proposition 29-11 $\rangle$

$x \square \ulcorner y = 0$

$\iff$ $\quad\langle$ by the assumption and Theorem 34-1 $\rangle$

$\ulcorner x \square \ulcorner y = 0$

$\iff$ $\quad\langle$ by Proposition 22-8 $\rangle$

$\ulcorner x \sqcup \ulcorner y = 0$

$\iff$ $\quad\langle$ by (48) $\rangle$

$\ulcorner(x \sqcup y) = 0$

$\iff$ $\quad\langle$ by Proposition 29-11 $\rangle$

$x \sqcup y = 0$

5. This follows directly from Theorems 34-3 and 34-4.
6. The implication $\Longrightarrow$ follows by the typing of $\ulcorner$ (Definition 24). The other implication follows from Proposition 29-3.
7. This is a simple consequence of Definition 33 and the other parts of this theorem. $\square$

Since $\sqcup$-complementation and $\Box$-complementation are equivalent, we can simply say that an element x is the complement of y relative to 1. Because an element x and its complement belong to the Boolean algebra $B_{\mathcal{D}}$, the complement of x is unique. This justifies defining x as "the" complement of y instead of "a" complement of y in Definition 33.

4 Definition of Angelic Operators in DAD

Our goal in this section is to define angelic operators from demonic ones, as was done when going from the angelic to the demonic universe (Section 2). This is done in order to study transformations between KAD and DAD (Sections 5 and 6). We add a subscript D to the angelic operators defined here, to denote that they are defined by demonic expressions. We start with the angelic partial order $\leq_D$.

Definition 35 (Angelic refinement). *Let x, y be elements of a DAD. We say that $x \leq_D y$ when the following two properties are satisfied.*

$$\ulcorner y \sqsubseteq \ulcorner x \tag{50}$$

$$x \sqsubseteq \ulcorner x \Box y \tag{51}$$

Proposition 37 below states that $\leq_D$ is a partial order. Moreover, it gives a formula using demonic operators for the angelic supremum with respect to this partial order. In order to demonstrate this theorem, we need the following lemma.

Lemma 36. *The function*

$$f : A_{\mathcal{D}} \times A_{\mathcal{D}} \to A_{\mathcal{D}}$$
$$(x, y) \mapsto (x \sqcup y) \sqcap \neg\ulcorner y \Box x \sqcap \neg\ulcorner x \Box y$$

satisfies the following four properties for all $x, y, z \in A_{\mathcal{D}}$. Note that f is well defined by Corollary 30-12.

1. $\ulcorner f(x, y) = \ulcorner x \sqcap \ulcorner y$
2. $f(x, x) = x$
3. $f(x, y) = f(y, x)$
4. $f(x, f(y, z)) = f(f(x, y), z)$

Proof.

1.
$$\ulcorner f(x, y)$$
$$= \quad \langle \text{ by hypothesis } \rangle$$
$$\ulcorner((x \sqcup y) \sqcap \neg\ulcorner y \Box x \sqcap \neg\ulcorner x \Box y)$$
$$= \quad \langle \text{ by Corollary 30-14 } \rangle$$
$$\ulcorner(x \sqcup y) \sqcap \ulcorner(\neg\ulcorner y \Box x) \sqcap \ulcorner(\neg\ulcorner x \Box y)$$

$=$ ⟨ by (48), Proposition 22-8 and Proposition 29-4 ⟩

$\ulcorner x \Box \ulcorner y \sqcap \neg \ulcorner y \Box \ulcorner x \sqcap \neg \ulcorner x \Box \ulcorner y$

$=$ ⟨ by Corollaries 23 and 30-1, and Boolean algebra ⟩

$\ulcorner x \sqcap \ulcorner y$

2. $f(x, x)$

$=$ ⟨ by hypothesis ⟩

$(x \sqcup x) \sqcap \neg \ulcorner x \Box x \sqcap \neg \ulcorner x \Box x$

$=$ ⟨ by Proposition 29-9 and (25) ⟩

$(x \sqcup x) \sqcap 0 \sqcap 0$

$=$ ⟨ by Corollary 30-3 and (22) ⟩

x

3. $f(x, y)$

$=$ ⟨ by hypothesis ⟩

$(x \sqcup y) \sqcap \neg \ulcorner y \Box x \sqcap \neg \ulcorner x \Box y$

$=$ ⟨ by (21) and Corollaries 30-9 and 30-8, since
$\ulcorner(\neg \ulcorner y \Box x) \Box \ulcorner(\neg \ulcorner x \Box y) = \neg \ulcorner y \Box \ulcorner x \Box \neg \ulcorner x \Box \ulcorner y = 0$
by Propositions 29-4 and 22-9 ⟩

$(y \sqcup x) \sqcap \neg \ulcorner x \Box y \sqcap \neg \ulcorner y \Box x$

$=$ ⟨ by hypothesis ⟩

$f(y, x)$

4. We first show $x \sqcup t \Box y = t \Box (x \sqcup y)$ (true for all x, y and all tests t).

$x \sqcup t \Box y$

$=$ ⟨ by Propositions 29-13 and 29-4 ⟩

$\ulcorner x \Box t \Box \ulcorner y \Box (x \sqcup t \Box y)$

$=$ ⟨ by (27) and Propositions 22-7, 22-10 ⟩

$\ulcorner x \Box t \Box \ulcorner y \Box (x \sqcup y)$

$=$ ⟨ by Propositions 22-10 and 29-13 ⟩

$t \Box (x \sqcup y)$

The main derivation follows. It repeatedly invokes Corollaries 30-8 and 30-9. Using (48) and Propositions 22-8 and 29-4, it is easy to check the operands of the various $\sqcap$ operators are pairwise disjoint, so that the condition $\ulcorner x \Box \ulcorner y$ of Corollary 30-9 is satisfied. This is what allows permuting the operands.

$f(x, f(y, z))$

$=$ ⟨ by hypothesis and Lemma 36-1 ⟩

$(x \sqcup ((y \sqcup z) \sqcap \neg \ulcorner z \Box y \sqcap \neg \ulcorner y \Box z)) \sqcap$
$\neg(\ulcorner y \sqcap \ulcorner z) \Box x \ \sqcap \ \neg \ulcorner x \Box ((y \sqcup z) \sqcap \neg \ulcorner z \Box y \sqcap \neg \ulcorner y \Box z)$

$$
\begin{aligned}
= &\qquad \langle \text{ by Corollaries 30-13, 30-4, 23, 30-1, and Boolean algebra } \rangle \\
& (x \sqcup y \sqcup z) \sqcap (x \sqcup \neg\ulcorner z \mathbin{\square} y) \sqcap (x \sqcup \neg\ulcorner y \mathbin{\square} z) \sqcap \\
& \neg\ulcorner y \mathbin{\square} \neg\ulcorner z \mathbin{\square} x \sqcap \neg\ulcorner x \mathbin{\square} (y \sqcup z) \sqcap \neg\ulcorner x \mathbin{\square} \neg\ulcorner z \mathbin{\square} y \sqcap \neg\ulcorner x \mathbin{\square} \neg\ulcorner y \mathbin{\square} z \\
= &\qquad \langle \text{ see the previous derivation and Corollaries 30-8 30-9 } \rangle \\
& (x \sqcup y \sqcup z) \sqcap \neg\ulcorner z \mathbin{\square} (x \sqcup y) \sqcap \neg\ulcorner y \mathbin{\square} (x \sqcup z) \sqcap \neg\ulcorner x \mathbin{\square} (y \sqcup z) \sqcap \\
& \neg\ulcorner y \mathbin{\square} \neg\ulcorner z \mathbin{\square} x \sqcap \neg\ulcorner x \mathbin{\square} \neg\ulcorner z \mathbin{\square} y \sqcap \neg\ulcorner x \mathbin{\square} \neg\ulcorner y \mathbin{\square} z \\
= &\qquad \langle \text{ by (21), (48), Propositions 22-10 and 29-4,} \\
&\qquad\quad \text{Corollaries 30-8 and 30-9, and Boolean algebra } \rangle \\
& (z \sqcup x \sqcup y) \sqcap \neg\ulcorner y \mathbin{\square} (z \sqcup x) \sqcap \neg\ulcorner x \mathbin{\square} (z \sqcup y) \sqcap \neg\ulcorner z \mathbin{\square} (x \sqcup y) \sqcap \\
& \neg\ulcorner x \mathbin{\square} \neg\ulcorner y \mathbin{\square} z \sqcap \neg\ulcorner z \mathbin{\square} \neg\ulcorner y \mathbin{\square} x \sqcap \neg\ulcorner z \mathbin{\square} \neg\ulcorner x \mathbin{\square} y \\
= &\qquad \langle \text{ see the previous derivation and Corollaries 30-8, 30-9 } \rangle \\
& (z \sqcup x \sqcup y) \sqcap (z \sqcup \neg\ulcorner y \mathbin{\square} x) \sqcap (z \sqcup \neg\ulcorner x \mathbin{\square} y) \sqcap \neg\ulcorner x \mathbin{\square} \neg\ulcorner y \mathbin{\square} z \sqcap \\
& \neg\ulcorner z \mathbin{\square} (x \sqcup y) \sqcap \neg\ulcorner z \mathbin{\square} \neg\ulcorner y \mathbin{\square} x \sqcap \neg\ulcorner z \mathbin{\square} \neg\ulcorner x \mathbin{\square} y \\
= &\qquad \langle \text{ by Corollaries 30-13, 30-4, 23, 30-1 and Boolean algebra } \rangle \\
& (z \sqcup ((x \sqcup y) \sqcap \neg\ulcorner y \mathbin{\square} x \sqcap \neg\ulcorner x \mathbin{\square} y)) \sqcap \neg(\ulcorner x \sqcap \ulcorner y) \mathbin{\square} z \sqcap \\
& \neg\ulcorner z \mathbin{\square} ((x \sqcup y) \sqcap \neg\ulcorner y \mathbin{\square} x \sqcap \neg\ulcorner x \mathbin{\square} y) \\
= &\qquad \langle \text{ by hypothesis and Lemma 36-1 } \rangle \\
& f(z, f(x, y)) \\
= &\qquad \langle \text{ by Lemma 36-3 } \rangle \\
& f(f(x, y), z) \qquad \square
\end{aligned}
$$

Proposition 37 (Angelic choice). *The angelic refinement of Definition 35 satisfies the following three properties.*

1. *For all x, $0 \leq_D x$.*
2. *For all x, y,*
$$x \leq_D y \iff f(x, y) = y \ ,$$
where f is the function defined in Lemma 36.
3. *$\leq_D$ is a partial order. Letting $x +_D y$ denote the supremum of x and y with respect to $\leq_D$, we have*
$$x +_D y = f(x, y) \ .$$

Proof.

1. Let x be any element of a DAD. From Proposition 29-11, we have $\ulcorner 0 = 0$, hence $\ulcorner x \sqsubseteq \ulcorner 0$. Also, $\ulcorner 0 \mathbin{\square} x = 0$, so $0 \sqsubseteq \ulcorner 0 \mathbin{\square} x$. The last two refinements are those from Definition 35, so $0 \leq_D x$.
2. $$
\begin{aligned}
& f(x, y) = y \\
\iff &\qquad \langle \text{ by Propositions 22-18, 29-2, 29-9 } \rangle \\
& \ulcorner y \mathbin{\square} f(x, y) = y \ \wedge \ \neg\ulcorner y \mathbin{\square} f(x, y) = 0 \\
\iff &\qquad \langle \text{ by Propositions 22-18, 29-2, and (25) } \rangle
\end{aligned}
$$

$$
\begin{array}{ll}
& \ulcorner x \Box \ulcorner y \Box f(x,y) = \ulcorner x \Box y \;\wedge\; \ulcorner x \Box \neg \ulcorner y \Box f(x,y) = 0 \;\wedge \\
& \neg \ulcorner x \Box \ulcorner y \Box f(x,y) = \neg \ulcorner x \Box y \;\wedge\; \neg \ulcorner x \Box \neg \ulcorner y \Box f(x,y) = 0 \\
\iff & \langle \text{ by definition of } f\text{, Corollaries 30-4, 30-3 and Propositions} \\
& \quad \text{22-9, 29-2, 29-9, 29-13 } \rangle \\
& \ulcorner x \Box \ulcorner y \Box (x \sqcup y) = \ulcorner x \Box y \;\wedge\; \neg \ulcorner y \Box x = 0 \;\wedge\; \neg \ulcorner x \Box y = \neg \ulcorner x \Box y \;\wedge\; 0 = 0 \\
\iff & \langle \text{ by Propositions 22-7 and 29-13 } \rangle \\
& \ulcorner x \Box (x \sqcup y) = \ulcorner x \Box y \;\wedge\; \neg \ulcorner y \Box x = 0 \\
\iff & \langle \text{ by (27), Proposition 29-2 and Proposition 29-11 } \rangle \\
& \langle \text{ by (27) and Propositions 29-2, 29-11 and 29-4 } \rangle \\
& x \sqcup \ulcorner x \Box y = \ulcorner x \Box y \;\wedge\; \neg \ulcorner y \Box \ulcorner x = 0 \\
\iff & \langle \text{ by Proposition 29-8 } \rangle \\
& x \sqcup \ulcorner x \Box y = \ulcorner x \Box y \;\wedge\; \ulcorner y \sqsubseteq \ulcorner x \\
\iff & \langle \text{ by (30) and by Definition 35 } \rangle \\
& x \leq_D y
\end{array}
$$

3. It follows from the previous point of the present theorem and by the fact that f is reflexive, symmetric and transitive (see Lemma 36). □

The following expected properties are a direct consequence of Lemma 36 and Proposition 37.

$$
\begin{aligned}
(x +_D y) +_D z &= x +_D (y +_D z) \\
x +_D y &= y +_D x \\
x +_D x &= x \\
0 +_D x &= x
\end{aligned}
$$

We now turn to the definition of angelic composition. But things are not as simple as for $\leq_D$ or $+_D$. The difficulty is due to the asymmetry between left and right caused by the difference between axioms (40) and (41), and by the absence of a codomain operator for "testing" the right-hand side of elements as can be done with the domain operator on the left. Consider the two relations

$$
Q = \{(0,0),(0,1),(1,2),(2,3)\} \qquad \text{and} \qquad R = \{(0,0),(2,2)\} \enspace .
$$

The angelic composition of Q and R is $Q \cdot R = \{(0,0),(1,2)\}$, while their demonic composition is $Q \Box R = \{(1,2)\}$. There is no way to express $Q \cdot R$ only in terms of $Q \Box R$. What we could try to do is to decompose Q as follows using the conditional

$$
Q = Q \Box \ulcorner R \sqcap Q \Box \neg \ulcorner R \sqcap (Q_1 \sqcup Q_2) \enspace ,
$$

where $Q_1 = \{(0,0)\}$ and $Q_2 = \{(0,1)\}$. Note that $Q \Box \ulcorner R = \{(1,2)\}$ and $Q \Box \neg \ulcorner R = \{(2,3)\}$, so that the domains of the three operands of $\sqcap$ are disjoint. The effect of $\sqcap$ is then just union. With these relations, it is possible to express the angelic composition as $Q \cdot R = Q \Box R \sqcap Q_1 \Box R$. Now, it is possible to extract $Q_1 \sqcup Q_2$

from Q, since $Q_1 \sqcup Q_2 = \neg\ulcorner(Q \square \ulcorner R) \square \neg\ulcorner(Q \square \neg\ulcorner R) \square Q$. The problem is that it is not possible to extract Q_1 from $Q_1 \sqcup Q_2$. On the one hand, Q_1 and Q_2 have the same domain; on the other hand, there is no test t such that $Q_1 = (Q_1 \sqcup Q_2) \square t$. This is what leads us to the following definition.

Definition 38. *Let t be a test. An element x of a DAD is said to be* t-decomposable *iff there are unique elements x_t and $x_{\neg t}$ such that*

$$
\begin{aligned}
x \;&=\; x \square t \sqcap x \square \neg t \sqcap (x_t \sqcup x_{\neg t}) \ , && (52)\\
\ulcorner x_t \;=\; \ulcorner x_{\neg t} \;&=\; \neg\ulcorner(x \square t) \square \neg\ulcorner(x \square \neg t) \square \ulcorner x \ , && (53)\\
x_t \;&=\; x_t \square t \ , && (54)\\
x_{\neg t} \;&=\; x_{\neg t} \square \neg t \ . && (55)
\end{aligned}
$$

And x is said to be decomposable *iff it is t-decomposable for all tests t.*

It is easy to see that all tests are decomposable. Indeed, the (unique) t-decomposition of a test s is

$$s = s \square t \sqcap s \square \neg t \sqcap (0 \sqcup 0) \ . \qquad (56)$$

Remark 39. The domains $\ulcorner(x \square t), \ulcorner(x \square \neg t)$ and $\ulcorner x_t$ (or $\ulcorner x_{\neg t}$) obtained by decomposing x as in Definition 38 are pairwise disjoint. That $\ulcorner x_t$ and $\ulcorner x_{\neg t}$ are disjoint from $\ulcorner(x \square t)$ and $\ulcorner(x \square \neg t)$ is obvious from (53). By Propositions 29-14, 22-9, (25) and Proposition 29-3,

$$\ulcorner(x \square t) \square \ulcorner(x \square \neg t) = \ulcorner(x \square t \square \neg t) = \ulcorner(x \square 0) = \ulcorner 0 = 0 \ ,$$

so that $\ulcorner(x \square t)$ and $\ulcorner(x \square \neg t)$ are disjoint as well. Moreover,

$$\ulcorner x = \ulcorner(x \square t) \sqcap \ulcorner(x \square \neg t) \sqcap \ulcorner x_t \ ,$$

since

$$
\begin{aligned}
& \ulcorner(x \square t) \sqcap \ulcorner(x \square \neg t) \sqcap \ulcorner x_t \\
= \quad & \langle \text{ by (53) } \rangle \\
& \ulcorner(x \square t) \sqcap \ulcorner(x \square \neg t) \sqcap \neg\ulcorner(x \square t) \square \neg\ulcorner(x \square \neg t) \square \ulcorner x \\
= \quad & \langle \text{ by Boolean algebra } \rangle \\
& \ulcorner(x \square t) \sqcap \ulcorner(x \square \neg t) \sqcap \ulcorner x \\
= \quad & \langle \text{ by Proposition 29-10 } \rangle \\
& \ulcorner x \ .
\end{aligned}
$$

This disjointness is often used in applications of Corollaries 30-8, 30-9 and 30-15.

One may wonder whether there exists a DAD with non-decomposable elements. The answer is yes. The following nine relations constitute such a DAD, with the operations given (they are the standard demonic operations on relations), omitting $\sqcap_\bullet$. The set of tests is $\{\mathsf{0}, \mathsf{s}, \mathsf{t}, \mathsf{1}\}$.

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathsf{s} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathsf{t} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathsf{a} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathsf{b} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathsf{c} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \mathsf{d} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \mathsf{e} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

$\sqcup$	0	s	t	1	a	b	c	d	e
0	0	0	0	0	0	0	0	0	0
s	0	s	0	s	s	d	d	d	0
t	0	0	t	t	e	t	e	0	e
1	0	s	t	1	a	b	c	d	e
a	0	s	e	a	a	c	c	d	e
b	0	d	t	b	c	b	c	d	e
c	0	d	e	c	c	c	c	d	e
d	0	d	0	d	d	d	d	d	0
e	0	0	e	e	e	e	e	0	e

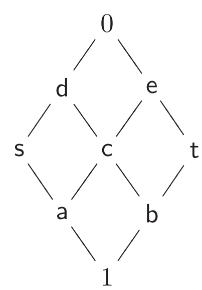
$\Box$	0	s	t	1	a	b	c	d	e
0	0	0	0	0	0	0	0	0	0
s	0	s	0	s	s	d	d	d	0
t	0	0	t	t	e	t	e	0	e
1	0	s	t	1	a	b	c	d	e
a	0	s	0	a	a	c	c	d	0
b	0	0	t	b	c	b	c	0	e
c	0	0	0	c	c	c	c	0	0
d	0	0	0	d	d	d	d	0	0
e	0	0	0	e	e	e	e	0	0

	$\times$
0	0
s	s
t	t
1	1
a	a
b	b
c	c
d	0
e	0

	$\ulcorner$
0	0
s	s
t	t
1	1
a	1
b	1
c	1
d	s
e	t

	$\neg$
0	1
s	t
t	s
1	0

The demonic refinement ordering corresponding to $\sqcup$ is represented in the following semilattice.



The elements $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}$ and e are not decomposable. For instance, to decompose c with respect to s would require the existence of relations

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} ,$$

which are not there.

Definition 40 (Angelic composition). *Let x and y be elements of a DAD such that x is decomposable. Then the angelic composition $\cdot_D$ is defined by*

$$x \cdot_D y \;=\; x \Box y \sqcap x_{\ulcorner y} \Box y \ .$$

Proposition 41. *Let x, y, z be decomposable elements of a DAD. Then,*

1. $1 \cdot_D x = x \cdot_D 1 = x$,
2. $0 \cdot_D x = x \cdot_D 0 = 0$,

3. $\ulcorner(x \cdot_D y) = \ulcorner x \square \neg \ulcorner(x \square \neg \ulcorner y)$
4. $\ulcorner(x \cdot_D (y \cdot_D z)) = \ulcorner((x \cdot_D y) \cdot_D z)$.

Proof.

1. Firstly, we show that $x_{\ulcorner 1} = 0$.

$$\ulcorner x_{\ulcorner 1}$$
$$= \quad \langle \text{ by (53) } \rangle$$
$$\ulcorner x_{\neg \ulcorner 1}$$
$$= \quad \langle \text{ by (55) } \rangle$$
$$\ulcorner x_{\neg \ulcorner 1} \square \neg \ulcorner 1$$
$$= \quad \langle \text{ by Propositions 29-3 and 22-11, and (25) } \rangle$$
$$0$$

So $x_{\ulcorner 1} = 0$ by Proposition 29-11. Here is the desired derivation.

$$1 \cdot_D x$$
$$= \quad \langle \text{ by Definition 40 } \rangle$$
$$1 \square x \sqcap 1_{\ulcorner x} \square x$$
$$= \quad \langle \text{ by (56) } \rangle$$
$$1 \square x \sqcap 0 \square x$$
$$= \quad \langle \text{ by (26) and (25) and Corollary 30-3 } \rangle$$
$$x$$
$$= \quad \langle \text{ by (26) and Corollary 30-3 } \rangle$$
$$x \square 1 \sqcap 0 \square 1$$
$$= \quad \langle \text{ see the previous derivation } \rangle$$
$$x \square 1 \sqcap x_{\ulcorner 1} \square 1$$
$$= \quad \langle \text{ by Definition 40 } \rangle$$
$$x \cdot_D 1$$

2. Firstly, we show that $x_{\ulcorner 0} = 0$.

$$\ulcorner x_{\ulcorner 0}$$
$$= \quad \langle \text{ by (54) } \rangle$$
$$\ulcorner x_{\ulcorner 0} \square \ulcorner 0$$
$$= \quad \langle \text{ by Proposition 29-3 and (25) } \rangle$$
$$0$$

So $x_{\ulcorner 0} = 0$ by Proposition 29-11. Here is the desired derivation.

$$
\begin{array}{ll}
& 0 \cdot_D x \\
= & \qquad \langle \text{ by Definition 40 } \rangle \\
& 0 \Box x \;\overline{\sqcap}\; 0_{\ulcorner x} \Box x \\
= & \qquad \langle \text{ by (56) } \rangle \\
& 0 \Box x \;\overline{\sqcap}\; 0 \Box x \\
= & \qquad \langle \text{ by (25) and Corollary 30-10 } \rangle \\
& 0 \\
= & \qquad \langle \text{ by (25) and Corollary 30-10 } \rangle \\
& x \Box 0 \;\overline{\sqcap}\; 0 \Box 0 \\
= & \qquad \langle \text{ see the previous derivation } \rangle \\
& x \Box 0 \;\overline{\sqcap}\; x_{\ulcorner 0} \Box 0 \\
= & \qquad \langle \text{ by Definition 40 } \rangle \\
& x \cdot_D 0
\end{array}
$$

3. Firstly, we need the following.

$$
\begin{array}{ll}
& \neg \ulcorner (x \Box \neg \ulcorner y) \Box \ulcorner x \sqsubseteq \ulcorner (x \Box \ulcorner y) \\
\Longleftrightarrow & \qquad \langle \text{ by Proposition 22-8 and Boolean algebra } \rangle \\
& \neg \ulcorner (x \Box \neg \ulcorner y) \sqsubseteq \ulcorner (x \Box \ulcorner y) \;\wedge\; \ulcorner x \sqsubseteq \ulcorner (x \Box \ulcorner y) \\
\Longleftrightarrow & \qquad \langle \text{ by Corollary 23, shunting and Proposition 29-10 } \rangle \\
& 0 \sqsubseteq \ulcorner (x \Box \neg \ulcorner y) \sqcup \ulcorner (x \Box \ulcorner y) \;\wedge\; \mathsf{true} \\
\Longleftrightarrow & \qquad \langle \text{ by Propositions 22-8 and 29-14 } \rangle \\
& 0 \sqsubseteq \ulcorner (x \Box \ulcorner y \Box \neg \ulcorner y) \\
\Longleftrightarrow & \qquad \langle \text{ by Proposition 22-9 } \rangle \\
& 0 \sqsubseteq \ulcorner 0 \\
\Longleftrightarrow & \qquad \langle \text{ by Proposition 29-11 } \rangle \\
& \mathsf{true}
\end{array}
$$

We are now ready to calculate the desired result.

$$
\begin{array}{ll}
& \ulcorner (x \cdot_D y) \\
= & \qquad \langle \text{ by Definition 40, Corollary 30-14 and (47) } \rangle \\
& \ulcorner (x \Box \ulcorner y) \;\overline{\sqcap}\; \ulcorner (x_{\ulcorner y} \Box \ulcorner y) \\
= & \qquad \langle \text{ by (54) } \rangle \\
& \ulcorner (x \Box \ulcorner y) \;\overline{\sqcap}\; \ulcorner x_{\ulcorner y} \\
= & \qquad \langle \text{ by (53) } \rangle \\
& \ulcorner (x \Box \ulcorner y) \;\overline{\sqcap}\; \neg \ulcorner (x \Box \ulcorner y) \Box \neg \ulcorner (x \Box \neg \ulcorner y) \Box \ulcorner x \\
= & \qquad \langle \text{ by Corollary 23 and 30-1 and Boolean algebra } \rangle
\end{array}
$$

$\ulcorner(x \Box \ulcorner y) \sqcap \neg\ulcorner(x \Box \neg\ulcorner y) \Box \ulcorner x$

$=$ ⟨ by the previous derivation and Boolean algebra ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \neg\ulcorner y)$

4. $\ulcorner(x \cdot_D (y \cdot_D z))$

$=$ ⟨ by Proposition 41-3 ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \neg(\ulcorner y \Box \neg\ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by De Morgan ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by Definition 38 ($\ulcorner y$-decomposition of x ⟩

$\ulcorner x \Box \neg\ulcorner((x \Box \ulcorner y \sqcap x \Box \neg\ulcorner y \sqcap (x_{\ulcorner y} \sqcup x_{\neg\ulcorner y})) \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by Corollary 30-15 and Remark 39 ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \ulcorner y \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)) \sqcap x \Box \neg\ulcorner y \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)) \sqcap$
$(x_{\ulcorner y} \sqcup x_{\neg\ulcorner y}) \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by Corollaries 30-7, 30-6 and Proposition 22-7 ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \ulcorner y \Box \ulcorner(y \Box \neg\ulcorner z) \sqcap x \Box \neg\ulcorner y \sqcap (x_{\ulcorner y} \sqcup x_{\neg\ulcorner y}) \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by Propositions 22-8 and 29-10 ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \ulcorner(y \Box \neg\ulcorner z) \sqcap x \Box \neg\ulcorner y \sqcap (x_{\ulcorner y} \sqcup x_{\neg\ulcorner y}) \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by Corollary 30-14 and (47) ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box y \Box \neg\ulcorner z \sqcap x \Box \neg\ulcorner y \sqcap (x_{\ulcorner y} \sqcup x_{\neg\ulcorner y}) \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by (54), (55) and (28) ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box y \Box \neg\ulcorner z \sqcap x \Box \neg\ulcorner y \sqcap$
$(x_{\ulcorner y} \Box \ulcorner y \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z)) \sqcup x_{\neg\ulcorner y} \Box \neg\ulcorner y \Box (\neg\ulcorner y \sqcap \ulcorner(y \Box \neg\ulcorner z))))$

$=$ ⟨ by Corollaries 30-7, 30-6 and Proposition 22-7 ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box y \Box \neg\ulcorner z \sqcap x \Box \neg\ulcorner y \sqcap (x_{\ulcorner y} \Box \ulcorner y \Box \ulcorner(y \Box \neg\ulcorner z) \sqcup x_{\neg\ulcorner y} \Box \neg\ulcorner y))$

$=$ ⟨ by (55) ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box y \Box \neg\ulcorner z \sqcap x \Box \neg\ulcorner y \sqcap (x_{\ulcorner y} \Box \ulcorner(y \Box \neg\ulcorner z) \sqcup x_{\neg\ulcorner y}))$

$=$ ⟨ by Corollary 30-14 and (48) ⟩

$\ulcorner x \Box \neg(\ulcorner(x \Box y \Box \neg\ulcorner z) \sqcap \ulcorner(x \Box \neg\ulcorner y) \sqcap (\ulcorner(x_{\ulcorner y} \Box \ulcorner(y \Box \neg\ulcorner z)) \sqcup \ulcorner x_{\neg\ulcorner y}))$

$=$ ⟨ since $\ulcorner x_{\neg\ulcorner y} = \ulcorner x_{\ulcorner y} \sqsubseteq \ulcorner(x_{\ulcorner y} \Box \ulcorner(y \Box \neg\ulcorner z))$ by (53) and Proposition 29-10 ⟩

$\ulcorner x \Box \neg(\ulcorner(x \Box y \Box \neg\ulcorner z) \sqcap \ulcorner(x \Box \neg\ulcorner y) \sqcap \ulcorner(x_{\ulcorner y} \Box \ulcorner(y \Box \neg\ulcorner z)))$

$=$ ⟨ by Boolean algebra and (47) ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \neg\ulcorner y) \Box \neg\ulcorner(x \Box y \Box \neg\ulcorner z) \Box \neg\ulcorner(x_{\ulcorner y} \Box y \Box \neg\ulcorner z)$

$=$ ⟨ by Boolean algebra and Corollary 30-14 ⟩

$\ulcorner x \Box \neg\ulcorner(x \Box \neg\ulcorner y) \Box \neg\ulcorner(x \Box y \Box \neg\ulcorner z \sqcap x_{\ulcorner y} \Box y \Box \neg\ulcorner z)$

$=$ ⟨ by Corollary 30-15 and Remark 39 ⟩

$$
\begin{array}{ll}
 & \ulcorner\!\ulcorner x \Box \neg \ulcorner\!\ulcorner(x \Box \neg \ulcorner\!\ulcorner y) \Box \neg \ulcorner\!\ulcorner((x \Box y \sqcap x_{\ulcorner\!\ulcorner y} \Box y) \Box \neg \ulcorner\!\ulcorner z) \\
= & \qquad \langle \text{ by Proposition 41-3 and Definition 40 } \rangle \\
 & \ulcorner\!\ulcorner((x \cdot_{D} y) \cdot_{D} z)
\end{array}
$$
□

We have not yet been able to show the associativity of $\cdot_D$ nor its distributivity over $+_D$.

By Definition 7 and Proposition 11-4,

$$\ulcorner(x \Box_A y) = \ulcorner x \cdot \neg \ulcorner(x \cdot \neg \ulcorner y) \ .$$

Comparing this expression with the one given in Proposition 41-3, namely

$$\ulcorner\!\ulcorner(x \cdot_D y) = \ulcorner\!\ulcorner x \Box \neg \ulcorner\!\ulcorner(x \Box \neg \ulcorner\!\ulcorner y) \ ,$$

reveals a nice duality. It remains to be seen whether this is accidental or whether there is something profound hiding there.

The last angelic operator that we define here is the iteration operator that corresponds to the Kleene star.

Definition 42 (Angelic iteration). *Let x be an element of a DAD. The* angelic finite iteration operator ^{*_D} *is defined by*

$$x^{*_D} = (x \sqcap 1)^{\times} \sqcup 1 \ .$$

Although we are still struggling to ascertain the properties of $\cdot_D$ (and, as a side effect, those of ^{*_D}), we have a conjecture that most probably holds. At least, it holds for a very important case (see Section 5).

Conjecture 43.

1. The set of decomposable elements of a DAD $\mathcal{D}$ is a subalgebra of $\mathcal{D}$.
2. For the subalgebra of decomposable elements of $\mathcal{D}$, the composition $\cdot_D$ is associative and distributes over $+_D$ (properties (5), (8) and (9)).
3. For the subalgebra of decomposable elements of $\mathcal{D}$, the iteration operator ^{*_D} satisfies the unfolding and induction laws of the Kleene star (properties (10), (14), (12) and (13)).

5 From KAD to DAD and Back

In this section, we introduce two transformations between the angelic and demonic worlds. The ultimate goal is to show how KAD and DAD are related one to the other.

Definition 44. *Let $(K, \mathsf{test}(K), +, \cdot, {}^*, 0, 1, \neg, \ulcorner\ )$ be a KAD. Let $\mathcal{F}$ denote the transformation that sends it to*

$$(K, \mathsf{test}(K), \sqcup_A, \Box_A, {}^{\times_A}, 0, 1, \sqcap_{A\bullet}, \ulcorner\ ) \ ,$$

where $\sqcup_A, \scriptstyle\square\displaystyle_A, {}^{\times_A}$ and $\sqcap_{A\bullet}$ are the operators defined in Proposition 9 and Definitions 10, 12 and 14, respectively.

Similarly, let $(A_\mathcal{D}, B_\mathcal{D}, \sqcup, \scriptstyle\square\displaystyle, {}^\times, 0, 1, \sqcap_\bullet, {}^{\ulcorner\!\ulcorner})$ be a DAD. Denote by $\mathcal{G}$ the transformation that sends it to

$$(A_\mathcal{D}, B_\mathcal{D}, +_D, \cdot_D, {}^{*_D}, 0, 1, \neg_D, {}^{\ulcorner\!\ulcorner}) \ ,$$

where $+_D, \cdot_D, {}^{*_D}$ and $\neg_D$ are the operators defined in Proposition 37, Definitions 40 and 42, and (35), respectively (since no special notation was introduced in Definition 20 to distinguish DAT negation from KAT negation, we have added a subscript D to $\neg$ in order to avoid confusion in Theorem 46).

By this definition, the transformations $\mathcal{F}$ and $\mathcal{G}$ transport the domain operator and the negation operator unchanged between the angelic and demonic worlds. Indeed, it turns out that $\ulcorner x = {}^{\ulcorner\!\ulcorner}x$ and $\neg t = \neg_D t$ are the right transformations.

Having defined $\mathcal{F}$ and $\mathcal{G}$, we can now state an important theorem. Just before, we need to introduce the following lemma.

Lemma 45. *Let K be a KAD. For all $x \in K$ and all $t \in \mathsf{test}(K)$,*

$$x = x \scriptstyle\square\displaystyle_A t \iff x = x \cdot t \ .$$

Proof.

$$\begin{aligned}
& x \scriptstyle\square\displaystyle_A t = x \\
\iff & \qquad \langle \text{ by Definition 10 } \rangle \\
& (x \to t) \cdot x \cdot t = x \\
\iff & \qquad \langle \text{ by Proposition 6-3 } \rangle \\
& (x \to t) \cdot x \cdot t \cdot t = x \cdot t \ \wedge \ (x \to t) \cdot x \cdot t \cdot \neg t = x \cdot \neg t \\
\iff & \qquad \langle \text{ by Definition 7, and Propositions 6-9 and 6-2 } \rangle \\
& \neg\ulcorner(x \cdot \neg t) \cdot x \cdot t \cdot t = x \cdot t \ \wedge \ 0 = x \cdot \neg t \\
\iff & \qquad \langle \text{ substituting 0 for } x \cdot \neg t \text{ in } \neg\ulcorner(x \cdot \neg t) \\
& \qquad\ \ \text{and by Propositions 6-9 and 6-2 } \rangle \\
& x \cdot t \cdot t = x \cdot t \ \wedge \ x \cdot t \cdot \neg t = x \cdot \neg t \\
\iff & \qquad \langle \text{ by Proposition 6-3 } \rangle \\
& x \cdot t = x
\end{aligned}$$

□

Theorem 46. *Let $\mathcal{K} = (K, \mathsf{test}(K), +, \cdot, {}^*, 0, 1, \neg, {}^\ulcorner)$ be a KAD and let $\mathcal{F}$ and $\mathcal{G}$ be the transformations introduced in Definition 44.*

1. *$\mathcal{F}(\mathcal{K})$ is a DAD.*
2. *All elements of $\mathcal{F}(\mathcal{K})$ are decomposable and, for $x \in K$ and $t \in \mathsf{test}(K)$,*

$$\begin{aligned}
x_t &= \ulcorner(x \cdot \neg t) \cdot x \cdot t \ , \\
x_{\neg t} &= \ulcorner(x \cdot t) \cdot x \cdot \neg t \ .
\end{aligned}$$

3. *$\mathcal{G}\circ\mathcal{F}$ is the identity on K. In other words, the algebra $(K, \mathsf{test}(K), +_D, \cdot_D, {}^{*_D}, 0, 1, \neg_D, {}^{\ulcorner})$ derived from the DAD $\mathcal{F}(\mathcal{K})$ is isomorphic to $\mathcal{K}$ (only the symbols denoting the operators differ).*
4. *Let $\mathcal{D}$ be a DAD. If ϕ is an isomorphism between $\mathcal{F}(\mathcal{K})$ and $\mathcal{D}$, then ϕ is also an isomorphism between $\mathcal{K}$ and $\mathcal{G}(\mathcal{D})$.*

Proof.

1. That $\mathcal{F}(\mathcal{K})$ is a DAD is just a compact restatement of Theorems 15, 16 and 17.
2. Let x be any element of K and t be any test. We have to show

$$x = x \mathbin{\square_A} t \mathbin{\overline{\sqcap}_A} x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t}) \ ,$$

where x_t and $x_{\neg t}$ have the unique solution given in the statement if they satisfy (53), (54) and (55). Remark 39 shows that $\ulcorner x$ can be split in three disjoint parts, namely $\ulcorner(x \mathbin{\square_A} t)$, $\ulcorner(x \mathbin{\square_A} \neg t)$ and $\ulcorner x_t$. Thus, by Proposition 22-18, the above equality holds if and only iff the following four equalities also do.

$$\begin{aligned}
\neg\ulcorner x \mathbin{\square_A} x &= \neg\ulcorner x \mathbin{\square_A} \big(x \mathbin{\square_A} t \mathbin{\overline{\sqcap}_A} x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\big)\\
\ulcorner(x \mathbin{\square_A} t) \mathbin{\square_A} x &= \ulcorner(x \mathbin{\square_A} t) \mathbin{\square_A} \big(x \mathbin{\square_A} t \mathbin{\overline{\sqcap}_A} x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\big)\\
\ulcorner(x \mathbin{\square_A} \neg t) \mathbin{\square_A} x &= \ulcorner(x \mathbin{\square_A} \neg t) \mathbin{\square_A} \big(x \mathbin{\square_A} t \mathbin{\overline{\sqcap}_A} x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\big)\\
\ulcorner x_t \mathbin{\square_A} x &= \ulcorner x_t \mathbin{\square_A} \big(x \mathbin{\square_A} t \mathbin{\overline{\sqcap}_A} x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\big)
\end{aligned}$$

Using Propositions 29-9 and 29-13, Corollary 30-4 and (53), the first equality reduces to $0 = 0$. The second one follows from Corollary 30-6, Proposition 29-2 and (46), and the third one from Remark 39, Propositions 30-8, 30-9, Corollary 30-6, Proposition 29-2 and (46). The following derivation is about the fourth equality and constructs the unique expressions for x_t and $x_{\neg t}$, assuming that x_t and $x_{\neg t}$ satisfy (53), (54) and (55). Uniqueness is due to the sequence of equivalences.

$$\begin{aligned}
&\ulcorner x_t \mathbin{\square_A} x = \ulcorner x_t \mathbin{\square_A} \big(x \mathbin{\square_A} t \mathbin{\overline{\sqcap}_A} x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\big)\\
\iff &\qquad \langle \text{ by Corollary 30-7, (53) and Boolean algebra } \rangle\\
&\ulcorner x_t \mathbin{\square_A} x = \ulcorner x_t \mathbin{\square_A} \big(x \mathbin{\square_A} \neg t \mathbin{\overline{\sqcap}_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\big)\\
\iff &\qquad \langle \text{ by Corollary 30-7, (53) and Boolean algebra } \rangle\\
&\ulcorner x_t \mathbin{\square_A} x = \ulcorner x_t \mathbin{\square_A} (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\\
\iff &\qquad \langle \text{ by Proposition 11-2 } \rangle\\
&\ulcorner x_t \cdot x = \ulcorner x_t \cdot (x_t \mathbin{\underline{\sqcup}_A} x_{\neg t})\\
\iff &\qquad \langle \text{ by Propositions 9 and 6-1, and (53) } \rangle\\
&\ulcorner x_t \cdot x = \ulcorner x_t \cdot (x_t + x_{\neg t})\\
\iff &\qquad \langle \text{ by (8), Proposition 6-5 and (53) } \rangle\\
&\ulcorner x_t \cdot x = x_t + x_{\neg t}
\end{aligned}$$

$\iff \qquad \langle$ by Proposition 6-3 $\rangle$

$\ulcorner x_t \cdot x \cdot t = (x_t + x_{\neg t}) \cdot t \ \wedge\ \ulcorner x_t \cdot x \cdot \neg t = (x_t + x_{\neg t}) \cdot \neg t$

$\iff \qquad \langle$ by (54), (55) and Lemma 45 $\rangle$

$\ulcorner x_t \cdot x \cdot t = (x_t \cdot t + x_{\neg t} \cdot \neg t) \cdot t \ \wedge\ \ulcorner x_t \cdot x \cdot \neg t = (x_t \cdot t + x_{\neg t} \cdot \neg t) \cdot \neg t$

$\iff \qquad \langle$ by (9), Propositions 6-1 and 6-2, (6) and (4) $\rangle$

$\ulcorner x_t \cdot x \cdot t = x_t \cdot t \ \wedge\ \ulcorner x_t \cdot x \cdot \neg t = x_{\neg t} \cdot \neg t$

$\iff \qquad \langle$ by (54), (55) and Lemma 45 $\rangle$

$\ulcorner x_t \cdot x \cdot t = x_t \ \wedge\ \ulcorner x_t \cdot x \cdot \neg t = x_{\neg t}$

$\iff \qquad \langle$ by (53), Proposition 11-4 and Boolean algebra $\rangle$

$(\neg(x \to t) + \neg \ulcorner x) \cdot (\neg(x \to \neg t) + \neg \ulcorner x) \cdot x \cdot t = x_t \ \wedge$
$(\neg(x \to t) + \neg \ulcorner x) \cdot (\neg(x \to \neg t) + \neg \ulcorner x) \cdot x \cdot \neg t = x_{\neg t}$

$\iff \qquad \langle$ by Boolean algebra, (9), Proposition 6-8 and (4) $\rangle$

$x_t = \neg(x \to t) \cdot \neg(x \to \neg t) \cdot x \cdot t \ \wedge$
$x_{\neg t} = \neg(x \to t) \cdot \neg(x \to \neg t) \cdot x \cdot \neg t$

$\iff \qquad \langle$ by Definition 7, Boolean algebra and Proposition 6-5 $\rangle$

$x_t = \ulcorner(x \cdot \neg t) \cdot x \cdot t \ \wedge\ x_{\neg t} = \ulcorner(x \cdot t) \cdot x \cdot \neg t$

3. To show this third point, it suffices to prove $x + y = x +_D y$, $x \cdot y = x \cdot_D y$, $x^* = x^{*_D}$ and $\neg t = \neg_D t$.

   (a) Firstly, we show that $x \leq y \iff x \leq_D y$.

   $x \leq_D y$

   $\iff \qquad \langle$ by Definition 35 $\rangle$

   $\ulcorner y \sqsubseteq_A \ulcorner x \ \wedge\ x \sqsubseteq_A \ulcorner x \square_A y$

   $\iff \qquad \langle$ by Definition 8 $\rangle$

   $\ulcorner x \leq \ulcorner y \ \wedge\ \ulcorner x \cdot \ulcorner y \leq \ulcorner x \ \wedge\ \ulcorner(\ulcorner x \square_A y) \leq \ulcorner x \ \wedge\ \ulcorner(\ulcorner x \square_A y) \cdot x \leq \ulcorner x \square_A y$

   $\iff \qquad \langle$ because $\ulcorner x \cdot \ulcorner y \leq \ulcorner x$ and by Proposition 11-2 $\rangle$

   $\ulcorner x \leq \ulcorner y \ \wedge\ \ulcorner(\ulcorner x \cdot y) \leq \ulcorner x \ \wedge\ \ulcorner(\ulcorner x \cdot y) \cdot x \leq \ulcorner x \cdot y$

   $\iff \qquad \langle$ (16) and Proposition 6-10 $\rangle$

   $\ulcorner x \leq \ulcorner y \ \wedge\ \ulcorner x \cdot \ulcorner y \cdot x \leq \ulcorner x \cdot y$

   $\iff \qquad \langle$ because $\ulcorner x \leq \ulcorner y$ $\rangle$

   $\ulcorner x \leq \ulcorner y \ \wedge\ \ulcorner x \cdot x \leq \ulcorner x \cdot y$

   $\iff \qquad \langle$ by monotonicity of $\ulcorner$ and $\cdot$ for $\Longleftarrow$ and
   by Proposition 6-5 and $x \leq \ulcorner x \cdot y \leq y$ for $\Longrightarrow$ $\rangle$

   $x \leq y$

   So $x + y = x +_D y$ by (11) and Proposition 37.

   (b) $x \cdot_D y$

   $= \qquad \langle$ by Definition 40 $\rangle$

$x \,\square_A\, y \sqcap_A x_{\ulcorner y} \,\square_A\, y$

$=$ ⟨ by Definition 49 and Definition 14 ⟩

$\ulcorner(x \,\square_A\, y) \,\square_A\, x \,\square_A\, y + \neg\ulcorner(x \,\square_A\, y) \,\square_A\, x_{\ulcorner y} \,\square_A\, y$

$=$ ⟨ by Proposition 29-2 and Proposition 11-2 ⟩

$x \,\square_A\, y + \neg\ulcorner(x \,\square_A\, y) \cdot (x_{\ulcorner y} \,\square_A\, y)$

$=$ ⟨ by Definition 10, Proposition 11-4 and Boolean algebra ⟩

$(x \to y) \cdot x \cdot y + (\neg(x \to y) + \neg\ulcorner x) \cdot (x_{\ulcorner y} \,\square_A\, y)$

$=$ ⟨ by (54) and Lemma 45 ⟩

$(x \to y) \cdot x \cdot y + (\neg(x \to y) + \neg\ulcorner x) \cdot x_{\ulcorner y} \cdot y$

$=$ ⟨ by Theorem 46-2 and Proposition 6-5 ⟩

$(x \to y) \cdot x \cdot y + (\neg(x \to y) + \neg\ulcorner x) \cdot \ulcorner(x \cdot \neg\ulcorner y) \cdot x \cdot y$

$=$ ⟨ by (9), Boolean algebra, Proposition 6-8, (6) and (4) ⟩

$(x \to y) \cdot x \cdot y + \neg(x \to y) \cdot \ulcorner(x \cdot \neg\ulcorner y) \cdot x \cdot y$

$=$ ⟨ by (9), Definition 7 and Boolean algebra ⟩

$((x \to y) + \neg(x \to y)) \cdot x \cdot y$

$=$ ⟨ by Boolean algebra and (7) ⟩

$x \cdot y$

(c) x^{*_D}

$=$ ⟨ by Definition 42 and (49) ⟩

$(x \sqcap_{A \ulcorner\ulcorner x} 1)^{\times_A} \sqcup_A 1$

$=$ ⟨ by Definition 14, Proposition 6-5 and (7) ⟩

$(x + \neg\ulcorner x)^{\times_A} \sqcup_A 1$

$=$ ⟨ by Definition 12 ⟩

$(x + \neg\ulcorner x)^* \,\square_A\, \ulcorner(x + \neg\ulcorner x) \sqcup_A 1$

$=$ ⟨ by Propositions 6-11 and 6-9, Boolean algebra and (26) ⟩

$(x + \neg\ulcorner x)^* \sqcup_A 1$

$=$ ⟨ by Proposition 9 ⟩

$\ulcorner((x + \neg\ulcorner x)^*) \cdot \ulcorner 1 \cdot ((x + \neg\ulcorner x)^* + 1)$

$=$ ⟨ by Propositions 6-14 and 6-9, and $1 \leq x^*$ by (10) ⟩

$(x + \neg\ulcorner x)^*$

$=$ ⟨ by the KA law $(x + y)^* = x^* \cdot (y \cdot x^*)^*$ ⟩

$x^* \cdot (\neg\ulcorner x \cdot x^*)^*$

$=$ ⟨ by (14), (8), Proposition 6-8, (6) and (7) ⟩

$x^* \cdot (\neg\ulcorner x)^*$

$$= \quad \langle \text{ for any test } t,\ t^* = 1 \ \rangle$$

$$x^*$$

(d) By (35) and Theorem 16-3, $\neg_D t = 0 \sqcap_{At} 1 = \neg t$.

4. Let $x, y \in K$ and $t \in \mathsf{test}(K)$. Because $\phi : \mathcal{F}(\mathcal{K}) \to \mathcal{D}$ is an isomorphism,

$$\begin{aligned} \phi(x \sqcup_A y) &= \phi(x) \sqcup \phi(y) \ , \\ \phi(x \,\square_A\, y) &= \phi(x) \square \phi(y) \ , \\ \phi(x^{\times_A}) &= (\phi(x))^{\times} \ , \\ \phi(0) &= 0 \ , \\ \phi(1) &= 1 \ , \\ \phi(x \sqcap_{At} y) &= \phi(x) \sqcap_{\phi(t)} \phi(y) \ , \\ \phi(\ulcorner x) &= \ulcorner(\phi(x)) \ , \\ \phi(\neg t) &= \neg(\phi(t)) \ . \end{aligned}$$

We have to show that $\phi : \mathcal{K} \to \mathcal{G}(\mathcal{D})$ is an isomorphism, that is,

$$\begin{aligned} \phi(x + y) &= \phi(x) +_D \phi(y) \ , \\ \phi(x \cdot y) &= \phi(x) \cdot_D \phi(y) \ , \\ \phi(x^*) &= (\phi(x))^{*_D} \ , \\ \phi(0) &= 0 \ , \\ \phi(1) &= 1 \ , \\ \phi(\ulcorner x) &= \ulcorner(\phi(x)) \ , \\ \phi(\neg t) &= \neg(\phi(t)) \ . \end{aligned}$$

We only show $\phi(x+y) = \phi(x) +_D \phi(y)$. The others are either trivial or proved similarly.

$$\begin{aligned} & \phi(x + y) \\ = \quad & \langle \text{ by Theorem 46-3 } \rangle \\ & \phi(x +_D y) \\ = \quad & \langle \text{ by Proposition 37 and the definition of } f \text{ in Lemma 36} \\ & \quad \text{(the demonic operators are those of } \mathcal{F}(\mathcal{K})) \ \rangle \\ & \phi\big((x \sqcup_A y) \sqcap_A \neg\ulcorner y \,\square_A\, x \sqcap_A \neg\ulcorner x \,\square_A\, y\big) \\ = \quad & \langle \ \phi : \mathcal{F}(\mathcal{K}) \to \mathcal{D} \text{ is an isomorphism } \rangle \\ & (\phi(x) \sqcup \phi(y)) \sqcap \neg\ulcorner(\phi(y)) \square \phi(x) \sqcap \neg\ulcorner(\phi(x)) \square \phi(y) \\ = \quad & \langle \text{ by Proposition 37 and the definition of } f \text{ in Lemma 36 } \rangle \\ & \phi(x) +_D \phi(y) \end{aligned}$$

□

Due to this theorem, the conjecture stated in the previous section holds for the DAD $\mathcal{F}(K)$. This is a very important case. Since the elements of $\mathcal{F}(K)$ are decomposable, this result gives much weight to the conjecture.

6 From DAD to KAD and Back

Let $\mathcal{D} = (A_{\mathcal{D}}, B_{\mathcal{D}}, \sqcup, \square, {}^{\times}, 0, 1, \sqcap_{\bullet}, {}^{\ulcorner\!\ulcorner})$ be a DAD. If $A_{\mathcal{D}}$ has non-decomposable elements, then $\mathcal{D}$ cannot be the image $\mathcal{F}(\mathcal{K})$ of a KAD $\mathcal{K}$, by Theorem 46-2. The question that is still not settled is whether the subalgebra $\mathcal{D}_d$ of decomposable elements of $\mathcal{D}$ is the image $\mathcal{F}(\mathcal{K})$ of some KAD $\mathcal{K}$. If Conjecture 43 holds, then this is the case and the composition of transformations $\mathcal{F} \circ \mathcal{G}$ is the identity on $\mathcal{D}_d$. This problem will be the subject of our future research.

7 Conclusion

The work on demonic algebra presented in this paper is just a beginning. Many avenues for future research are open. First and foremost, Conjecture 43 must be solved. In relation to this conjecture, the properties of non-decomposable elements are also intriguing. Are there concrete models useful for Computer Science where these elements play a rôle?

Another line of research is the precise relationship of DAD with the other refinement algebras and most particularly those of [16,24,25,27]. DAD has stronger axioms than these algebras, and thus these contain a DAD as a substructure. Some basic comparisons can already be done. For instance, DADs can be related to the *command algebras* of [16] as follows. Suppose a KAD $\mathcal{K} = (K, \mathsf{test}(K), +, \cdot, {}^*, 0, 1, \neg, {}^{\ulcorner})$. A *command* on $\mathcal{K}$ is an ordered pair (x, s), where $x \in K$ and $s \in \mathsf{test}(K)$. The test s denotes the "domain of termination" of x. If $s \leq {}^{\ulcorner}x$, the command (x, s) is said to be *feasible*; otherwise, it is *miraculous*. The set of non-miraculous commands of the form $(x, {}^{\ulcorner}x)$, with the appropriate definition of the operators, is isomorphic to the KAD-based demonic algebra $\mathcal{D}$ obtained from $\mathcal{K}$. If K is the set of all relations over a set S, then $\mathcal{D}$ is isomorphic to the non-miraculous conjunctive predicate transformers on S; this establishes a relationship with the refinement algebras of [25,27], which have predicate transformers as their main model. The algebras in [25,27] have two kinds of tests, *guards* and *assertions*. Assertions correspond to the tests of DAD and the *termination* operator τ of [25] corresponds to the domain operator of DAD.

Finally, let us mention the problem of infinite iteration. In DAD, there is no infinite iteration operator. One cannot be added by simply requiring it to be the greatest fixed point of $\lambda(z :: x \square_A z \sqcup_A 1)$, since this greatest fixed point is always 0. In [13], tests denoting the starting points of infinite iterations for an element x are obtained by using the greatest fixed point (in a KAD) of $\lambda(t :: {}^{\ulcorner}(x \cdot t))$. We intend to determine whether a similar technique can be used in DAD.

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